ON THE INVALIDITY OF FOURIER SERIES EXPANSIONS OF FRACTIONAL ORDER

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Abstract: The purpose of this short paper is to show the invalidity of a Fourier series expansion of fractional order as derived by G. Jumarie in a series of papers. In his work the exponential functions $e^{in\omega x}$ are replaced by the Mittag-Leffler functions $E_\alpha\left(i(n\omega x)^\alpha\right)$, over the interval $[0, M_\alpha/\omega]$ where $0 < \omega < \infty$ and M_α is the period of the function $E_\alpha\left(ix^\alpha\right)$, i.e., $E_\alpha\left(ix^\alpha\right) = E_\alpha\left(i(x + M_\alpha)^\alpha\right)$.

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1. INTRODUCTION

The Mittag-Leffler function which is named after the Swedish mathematician Mittag-Leffler, who introduced it in 1903 [17], has recently been the object of many research papers because it plays an important role in Fractional Calculus. It appears in solutions of some fractional differential and integral equations and as a result it has found some physical applications [14].

The Mittag-Leffler function is an entire function of order $1/\alpha, \alpha > 0$, defined by the power series

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k\alpha)}, \quad z \in \mathbb{C}.$$

It is easy to see that

$$E_1(z) = e^z, \quad E_2(z^2) = \cosh z.$$

Because the Mittag-Leffler function generalizes the exponential function, one wonders if it generalizes or at least shares some properties of the exponential function. Several people explored that idea, among them is G. Jumarie, who derived the following proposition [8]

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Proposition 1.1. *Suppose that $f : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and possesses for all $k \in \mathbb{N}$ Riemann-Liouville fractional derivatives of order $k\alpha$, with $0 < \alpha \leq 1$. Let $h > 0$. Then*

(1)
$$f(x + h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1 + k\alpha)} f^{(k\alpha)}(x),$$

where D^α is the modified Riemann-Liouville fractional derivative.

He used this proposition to show that

(2)
$$f(x + h) = E_\alpha(h^\alpha D^\alpha)f(x),$$

which he then employed to establish that

(3)
$$D^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha), \quad x \geq 0.$$

This equation is a generalization of the relation

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x},$$

and reduces to it when $\alpha = 1$. Moreover, he also showed that the semi-group property $e^{\lambda x} e^{\lambda y} = e^{\lambda(x+y)}$ may be extended to

(4)
$$E_\alpha(\lambda x^\alpha) E_\alpha(\lambda y^\alpha) = E_\alpha(\lambda(x + y)^\alpha), \quad 0 < \alpha \leq 1, \ \lambda \in \mathbb{C}.$$

In fact, in [12] he gave several proofs of this relation.

Analogous to the definition of $\cos x$ and $\sin x$ in terms of the exponential function, Jumarie defined $\cos_\alpha(x^\alpha)$ and $\sin_\alpha(x^\alpha)$ by

$$E_\alpha(ix^\alpha) = \cos_\alpha(x^\alpha) + i \sin_\alpha(x^\alpha), \quad x \geq 0.$$

which leads to

$$\cos_\alpha{}^2(x^\alpha) + \sin_\alpha{}^2(x^\alpha) = E_\alpha(ix^\alpha) E_\alpha(-ix^\alpha).$$

Therefore,

$$\cos_\alpha{}^2(x^\alpha) + \sin_\alpha{}^2(x^\alpha) = 1,$$

is equivalent to

$$E_\alpha(ix^\alpha) E_\alpha(-ix^\alpha) = 1,$$

that is

(5)
$$(E_\alpha(ix^\alpha))^{-1} = E_\alpha(-ix^\alpha).$$

But here we note that this relation does not follow from the functional equation

$$E_\alpha(\lambda x^\alpha) E_\alpha(\lambda y^\alpha) = E_\alpha(\lambda(x + y)^\alpha)$$

because if we put $y = -x$, we have

$$E_\alpha(\lambda x^\alpha) E_\alpha(\lambda(-x)^\alpha) = E_\alpha(\lambda(x - x)^\alpha) = E_\alpha(0) = 1$$

which implies that

(6)
$$(E_\alpha(\lambda x^\alpha))^{-1} = E_\alpha(\lambda(-x)^\alpha),$$

and when combined with (5) leads to

(7)
$$(E_\alpha(ix^\alpha))^{-1} = E_\alpha(i(-x)^\alpha) = E_\alpha(-ix^\alpha).$$

Under the assumption that there exists a real number $M_\alpha > 0$ such that $E_\alpha(iM_\alpha^\alpha)$ $= 1$, the periodicity of the function $E_\alpha(ix^\alpha)$, and hence, the periodicity of $\cos_\alpha(x^\alpha)$ and $\sin_\alpha(x^\alpha)$, will follow since

$$E_\alpha(ix^\alpha) = E_\alpha(ix^\alpha)E_\alpha(iM_\alpha^\alpha) = E_\alpha(i(x + M_\alpha))^\alpha).$$

The orthogonality of $\cos_\alpha(mx)^\alpha$ and $\sin_\alpha(mx)^\alpha$ now follows from the periodicity of these functions together with the fact that

$$\cos_\alpha{}^2(x^\alpha) + \sin_\alpha{}^2(x^\alpha) = 1.$$

It was also shown that $\cos_\alpha(mx)^\alpha$ and $\sin_\alpha(mx)^\alpha$ form an orthogonal basis for $L^2[0, M_\alpha]$. In [8], Sec. 6, he obtained an expansion in terms of fractional sine and cosine functions as follows. Let f be a periodic function with period M_α/ω. Then f can be expanded in a Fourier-type series of the form

$$f(x) = a_0/2 + \sum_{n=1}^\infty a_n \cos_\alpha(n\omega x)^\alpha + b_n \sin_\alpha(n\omega x)^\alpha,$$

where

$$a_n = 2\left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x)\cos_\alpha(n\omega x)^\alpha (dx)^\alpha,$$

and

$$b_n = 2\left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x)\sin_\alpha(n\omega x)^\alpha (dx)^\alpha.$$

Parseval's relation takes the form

$$\int_0^{M_\alpha/\omega} f^2(x)(dx)^\alpha = \frac{1}{2}\left(\frac{M_\alpha}{\omega}\right)^\alpha \sum_{n=0}^\infty \left(a_n^2 + b_n^2\right).$$

These results paved the way to developing a parallel approach to harmonic analysis based on these fractional cosine and sine functions.

Fascinated with these results, the authors of this article tried to derive new results using these fractional trigonometric functions and their orthogonality. However, they soon realized that there were some errors in Jumarie's derivations and some of the above relations cannot hold. The purpose of this paper is to show exactly that.

2. Preliminary Result

In this section we show by examples that some of Jumarie's results are invalid. First, we observe from Eq. (7) that we should have $E_\alpha(i(-x)^\alpha) = E_\alpha(-ix^\alpha)$. But from the definition

$$E_\alpha(\lambda x^\alpha) = \sum_{k=0}^\infty \frac{\lambda^k x^{\alpha k}}{\Gamma(1 + k\alpha)},$$

we have

$$E_\alpha(-ix^\alpha) = \sum_{k=0}^\infty \frac{(-1)^k i^k x^{\alpha k}}{\Gamma(1 + k\alpha)}$$

and

$$E_\alpha(i(-x)^\alpha) = \sum_{k=0}^\infty \frac{(-1)^{\alpha k} i^k x^{\alpha k}}{\Gamma(1 + k\alpha)},$$

and for the last two representations to be equal, we must have $(-1)^k = (-1)^{\alpha k}$ which happens only if $\alpha = 1$.

We can also show directly that $E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha) \neq 1$. For,

$$
\begin{aligned}
E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha) &= \left(\sum_{k=0}^{\infty} \frac{(ix^\alpha)^k}{\Gamma(1+\alpha k)}\right)\left(\sum_{m=0}^{\infty} \frac{(-ix^\alpha)^m}{\Gamma(1+\alpha m)}\right) \\
&= \left(\sum_{k,m=0}^{\infty} \frac{(-1)^m(ix^\alpha)^{k+m}}{\Gamma(1+\alpha m)\Gamma(1+\alpha k)}\right) \\
&= \sum_{n=0}^{\infty} i^n x^{\alpha n} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{\Gamma(1+\alpha k)\Gamma(1+\alpha(n-k))} \\
&= \sum_{n=0}^{\infty} i^n x^{\alpha n} A_n(\alpha),
\end{aligned}
$$

where

$$
A_n(\alpha) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{\Gamma(1+\alpha k)\Gamma(1+\alpha(n-k))}.
$$

It is easy to see that $A_0 = 1, A_{2k+1} = 0, \forall k \in \mathbb{N}_0$. However,

$$
\begin{aligned}
A_2 &= \frac{1}{\Gamma(1+2\alpha)} - \frac{1}{\Gamma^2(1+\alpha)} + \frac{1}{\Gamma(1+2\alpha)} \\
&= \frac{2}{\Gamma(1+2\alpha)} - \frac{1}{\Gamma^2(1+\alpha)} \\
&= \frac{2\Gamma^2(1+\alpha) - \Gamma(1+2\alpha)}{\Gamma(1+2\alpha)\Gamma^2(1+\alpha)} \neq 0.
\end{aligned}
$$

In fact, all the even $A_{2k} \neq 0$, but all of them are equal to zero only if $\alpha = 1$.

In Figure 1, the product $p(x, \alpha) := E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha)$ is plotted for several values of $0 < \alpha \leq 1$.
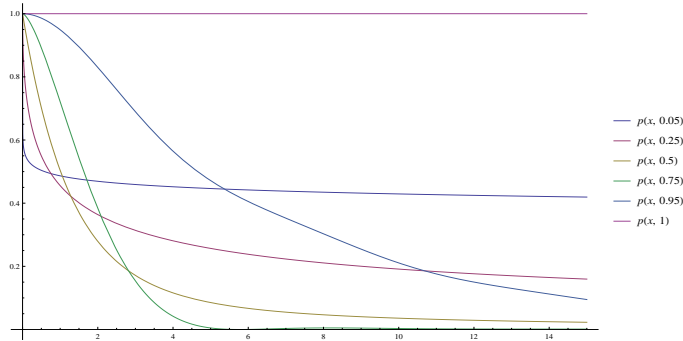

FIGURE 1. The product $E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha)$ for several values of $0 < \alpha \leq 1$.

Moreover, if there existed a real number $M > 0$ such that $E_\alpha(iM^\alpha) = 1$, we must have $\cos_\alpha(M^\alpha) = 1$, $\sin_\alpha(M^\alpha) = 0$. But for example, for $\alpha = \frac{1}{2}$, we have

$$\cos_{1/2}(x^{1/2}) = e^{-x} \text{ and } \sin_{1/2}(x^{1/2}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1/2}}{\Gamma(k + 3/2)}.$$

Using the Pochammer notation, we have $\Gamma(k + 3/2) = \Gamma(3/2)(3/2)_k$ and hence

$$
\begin{aligned}
\sin_{1/2}(x^{1/2}) &= \frac{x^{1/2}}{\Gamma(3/2)} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(3/2)_k} \\
&= \frac{x^{1/2}}{\Gamma(3/2)} \sum_{k=0}^{\infty} \frac{(1)_k(-x)^k}{(3/2)_k(1)_k} \\
&= \frac{x^{1/2}}{\Gamma(3/2)} \sum_{k=0}^{\infty} \frac{(1)_k(-x)^k}{(3/2)_k k!} = \frac{x^{1/2}}{\Gamma(3/2)} \Phi(1, 3/2; -x),
\end{aligned}
$$

where $\Phi(a, b; x)$ is the confluent hypergeometric function. Using the relation

$$\Phi(a, b; x) = e^x \Phi(b - a, b; -x)$$

we obtain

$$\Phi(1, 3/2; -x) = e^{-x} \Phi(1/2, 3/2; x),$$

or

$$\sin_{1/2}(x^{1/2}) = \frac{x^{1/2}}{\Gamma(3/2)} e^{-x} \Phi(1/2, 3/2; x).$$

But in view of the relationship between the confluent hypergeometric and the error function

$$\mathrm{Erf}(z) = z\Phi\left(1/2, 3/2; -z^2\right) = \int_0^z e^{-t^2}\, dt,$$

it is clear that the only solution for $\cos_{1/2}(M^{1/2}) = 1$ and $\sin_{1/2}(M^{1/2}) = 0$ is $M = 0$, i.e., there is no period for these fractional trigonometric functions for $\alpha = \frac{1}{2}$.

In the next section we show that our results hold not only for $\alpha = 1/2$ but for all $0 < \alpha < 1$.

3. The Main Result

In this section we extend the results of the previous section from $\alpha = 1/2$ to all $0 < \alpha < 1$. To this end, we need the following definition.

Definition 3.1. *A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called completely monotonic (CM) if $f \in C^\infty$ and $\forall n \in \mathbb{N}_0 \, \forall x \in \mathbb{R}^+ : (-1)^n f^{(n)}(x) \geq 0$.*

Theorem 3.1. *Let $0 \leq \alpha < 1$. Then there does not exist an $M_\alpha > 0$ so that*

$$E_\alpha(iM_\alpha^\alpha) = 1 \quad \text{or, equivalently,} \quad \cos_\alpha(M_\alpha^\alpha) = 1 \ \text{and} \ \sin_\alpha(M_\alpha^\alpha) = 0.$$

Moreover, the relation

$$(8) \qquad E_\alpha(\lambda x^\alpha) E_\alpha(\lambda y^\alpha) = E_\alpha(\lambda(x + y)^\alpha), \quad 0 < \alpha \leq 1, \ \lambda \in \mathbb{C},$$

can hold only if $\alpha = 1$.

Proof. We first note that $\cos_\alpha(x^\alpha) = E_{2\alpha}(-x^{2\alpha})$. This is verified by direct compu-
tation or by referring to the duplication formula [13, Eqn. 2.14]. It suffices to show
that $\cos_\alpha(x^\alpha) < 1$ for all $x > 0$.

To this end, we use a result from [18] which states that the Mittag-Leffler function
$E_\alpha(-x)$ is completely monotonic for all $0 \le \alpha \le 1$ and $x \ge 0$. As $x \mapsto x^\alpha$ is a
Bernstein function for $0 < \alpha < 1$ and the composition $E_\alpha(-\cdot) \circ (\cdot)^\alpha$ of the CM
function $E_\alpha(-\cdot)$ with the Bernstein function $x \mapsto x^\alpha$ is again a CM function [20,
Theorem 3.7. (ii)], we see that $E_{2\alpha}(-x^{2\alpha})$ is a CM function for $0 < \alpha < \frac{1}{2}$.

As $E_{2\alpha}(0) = 1$ and $E_{2\alpha}(-x^{2\alpha}) \to 0+$ (see, for instance, [15, Eq. (3.3)]), the
complete monotonicity for $0 < \alpha < \frac{1}{2}$ and $x > 0$ now implies that there cannot
exist an $M_\alpha > 0$ such that $\cos_\alpha(M_\alpha^\alpha) = 1$.

Now suppose $\frac{1}{2} < \alpha < 1$. It is shown in [13, Section 5.2] that in this case the
Mittag-Leffler function $E_{2\alpha}$ can be written as a sum of two functions, $f_{2\alpha}$ and $g_{2\alpha}$,
where the former is CM and the latter oscillatory. More precisely,

$$E_{2\alpha}(-x^{2\alpha}) = f_{2\alpha}(-x^{2\alpha}) + g_{2\alpha}(-x^{2\alpha}), \quad \frac{1}{2} < \alpha < 1, \ x \ge 0,$$

where

$$f_{2\alpha}(-x^{2\alpha}) := \frac{1}{\pi} \int_0^\infty \frac{e^{-sx^{2\alpha}} s^{2\alpha} \sin(2\alpha\pi)}{s^{4\alpha} + 2s^{2\alpha} \cos(2\alpha\pi) + 1} \, ds,$$

and

$$g_{2\alpha}(-x^{2\alpha}) := \frac{2}{\alpha} e^{-x^{2\alpha} \cos(\pi/2\alpha)} \cos\left(x^{2\alpha} \sin\frac{\pi}{2\alpha}\right).$$

The function $f_{2\alpha}$ is CM, satisfies $f_{2\alpha}(0) = 1 - \frac{2}{\alpha}$, and increases towards zero from
above, whereas the function $g_{2\alpha}$ is oscillatory with exponentially decaying amplitude
and $g_{2\alpha}(0) = \frac{2}{\alpha}$. Hence, $E_{2\alpha}(-x^{2\alpha}) < 1$, for all $\frac{1}{2} < \alpha < 1$ and $x > 0$. Thus, there
cannot exist an M_α in this case either.

The case $\alpha = \frac{1}{2}$ was considered above.

Next we show that the purported functional equation (8) can hold only if $\alpha = 1$.
First note that $E_\alpha(\lambda x^\alpha) = E_\alpha \circ h(x)$, where $h(x) := \lambda x^\alpha$. Writing F_α for the
composition $E_\alpha \circ h$, Eq. (8) reads

$$(9) \qquad\qquad\qquad F_\alpha(x) F_\alpha(y) = F_\alpha(x + y).$$

However, the only nonzero continuous solutions of Eq. (9) are exponential functions
of the form e^{cx}, where $c \in \mathbb{C}$; see for instance [1, Chapter 2]. Therefore, $F_\alpha(x) =
E_\alpha(\lambda x^\alpha) = e^{cx}$, for all x. Successively differentiating the power series for $E_\alpha(\lambda x^\alpha)$
and e^{cx} and letting $x \to 0$, shows that $\lambda = c$ and $\alpha = 1$. $\qquad\qquad\square$

Figure 2 below depicts some graphs of $\cos_\alpha(x^\alpha) = E_{2\alpha}(-x^{2\alpha})$, for $0 < x \le 1$.
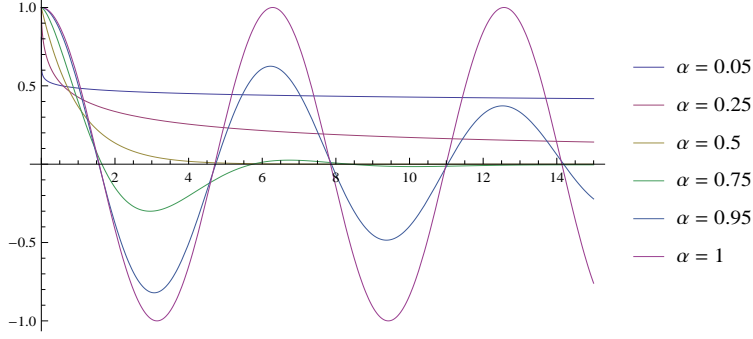The case $\alpha := 1$ produces the cosine function.

In [12], the purported functional equation for $E_\alpha(\lambda x^\alpha)$ is derived using several
approaches. One of them is the product rule

$$(10) \qquad\qquad\qquad D^\alpha(fg) = g D^\alpha f + f D^\alpha g.$$

However, this rule is not correct.

To see this, let $g := f := x^{1/2}$. It is straight-forward to establish that for $x \ge 0$,
$p > -1$, and $0 < \alpha < 1$, the fractional derivative of x^p is given by

$$D^\alpha x^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} x^{p-\alpha}.$$

FIGURE 2. $\cos_\alpha(x^\alpha)$ for several values of $0 < \alpha \leq 1$.

Hence, the left-hand side of (10) computes to

$$D^\alpha(x) = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)},$$

whereas the right-hand side equals

$$2x^{1/2}D^\alpha(x^{1/2}) = \frac{\sqrt{\pi}\, x^{1-\alpha}}{\Gamma(\frac{3}{2}-\alpha)}.$$

Both sides are identical only if $\alpha = 1$.

Similarly, the two purported chain rules, which are also employed in the derivation of (8), namely

$$D^\alpha(f \circ u)(x) = \frac{df}{du}\, D^\alpha u(x) \quad \text{and} \quad D^\alpha(f \circ u)(x) = (D_u^\alpha f) \cdot \left(\frac{du}{dx}\right)^\alpha,$$

are not correct. To validate this, take for the former, $f(x) := x^2$ and $u(x) := x^{1/2}$, $x \geq 0$. Then, for any $0 < \alpha \leq 1$, one has

$$(11) \qquad D^\alpha(f \circ u)(x) = D^\alpha x = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}$$

and

$$(12) \qquad \left.\frac{df}{du}\right|_{u=x^{1/2}} \cdot D^\alpha u(x) = 2x^{1/2}D^\alpha x^{1/2} = \frac{\sqrt{\pi}\, x^{1-\alpha}}{\Gamma(\frac{3}{2}-\alpha)},$$

and this two expressions are identical only if $\alpha = 1$. Now for the latter, take $f(u) := u^{1/2}$ and $u(x) := x^2$, $x \geq 0$. Then, for any $0 < \alpha \leq 1$, we again have
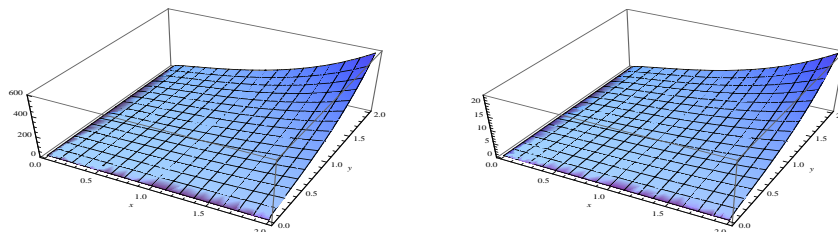
$$D^\alpha(f \circ u)(x) = D^\alpha x = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)},$$

whereas now

$$(13) \qquad \left.(D_u^\alpha f)\right|_{u=x^2} \cdot \left(\frac{du}{dx}\right)^\alpha = \left.D_u^\alpha(u^{1/2})\right|_{u=x^2}(2x)^\alpha = \frac{2^{\alpha-1}\sqrt{\pi}\, x^{1-\alpha}}{\Gamma(\frac{3}{2}-\alpha)}.$$

Eqns. (11), (12), and (13) are identical only if $\alpha = 1$.

For illustrative purposes, we plotted the difference $E_\alpha(\lambda(x+y)^\alpha) - E_\alpha(\lambda x^\alpha)E_\alpha(\lambda y^\alpha)$ in Figure 3 for $(x,y) \in [0,2]^2$, $\lambda = 1$, $\alpha = \frac{1}{4}$ (left), and $\alpha = \frac{3}{4}$ (right).

FIGURE 3. The difference $E_\alpha(\lambda(x+y)^\alpha) - E_\alpha(\lambda x^\alpha)E_\alpha(\lambda y^\alpha)$.

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