The Relation of Spatial and Tensor Product of Arveson Systems — The Random Set Point of View

Volkmar Liebscher

September 4, 2015,16:27:32

Abstract

We characterise the embedding of the spatial product of two Arveson systems into their tensor product using the random set technique. An important implication is that the spatial tensor product does not depend on the choice of the reference units, i.e. it is an intrinsic construction. There is a continuous range of examples coming from the zero sets of Bessel processes where the two products do not coincide. The lattice of all subsystems of the tensor product is analised in different cases. As a by-product, the Arveson systems coming from Bessel zeros prove to be primitive in the sense of [15].

1 Introduction

In a series of seminal papers in 1989 and 1990, ARVESON associated with every E_0-*semigroup* (a semigroup of unital endomorphisms) on $\mathscr{B}(H)$ its continuous product system of Hilbert spaces, *Arveson system* for short. Briefly, it is a measurable family of separable Hilbert spaces $\mathscr{E} = (\mathscr{E}_t)_{t\geq 0}$ with an associative identification

$$\mathscr{E}_s \otimes \mathscr{E}_t = \mathscr{E}_{s+t}, \qquad s,t \geq 0.$$

ARVESON showed in [3] that E_0-semigroups are classified by their Arveson system up to cocycle conjugacy. By a *spatial* Arveson system we understand a pair $(\mathscr{E},u)$ of an Arveson system $\mathscr{E}$ and a *normalised unit* u. The latter is a measurable section $u = (u_t)_{t\geq 0}$ of unit vectors $u_t \in \mathscr{E}_t$ that factor as

$$u_s \otimes u_t = u_{s+t}, \qquad s,t \geq 0.$$

For a thorough account on Arveson systems we refer to the monograph [4].

It is known that the structure of a spatial Arveson system $(\mathscr{E},u)$ depends on the choice of the reference unit $(u_t)_{t\geq 0}$. In fact, TSIRELSON [30] and MARKIEWICZ

AND POWERS [19] showed for example Arveson systems $(\mathscr{E}_t)_{t\geq 0}$ with normalised units $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$ that there does not exist an automorphism of $\mathscr{E}$ that sends $(u_t)_{t\geq 0}$ to $(v_t)_{t\geq 0}$. Thus we have to distinguish Arveson systems and spatial Arveson systems carefully.

The focus of the present paper is the spatial product of two spatial Arveson systems $(\mathscr{E},u)$ and $(\mathscr{F},v)$. It is, as a subsystem of $(\mathscr{E}_t\otimes\mathscr{F}_t)_{t\geq 0}$, formally given by

$$(\mathscr{E}_u\otimes_v\mathscr{F})_t = \lim_{(t_1,\ldots,t_n)\in\Pi^t}\bigotimes_{i=1}^{n}((u_t\otimes v_t^{\perp})\oplus(\mathbb{C}u_t\otimes v_t)\oplus(u_t^{\perp}\otimes v_t)). \qquad (1)$$

Here, the limit is taken over finer and finer partitions of $[0,t]$. This is exactly the description of the product system arising from Powers sum of E_0-semigroups, see [22, 8]. It also arises as a special case of inclusion systems [9]. For this structure, the two units u and v are glued together into one unit of the product.

Interestingly, [23] showed that a similar construction works for product systems of Hilbert *modules*, too. This was very important, since for general product systems of Hilbert modules, the fibrewise tensor product need not yield a product system. Unfortunately, the random set technique used below was not extended to the module situation yet. Thus, we deal here with Arveson systems only.

Not spatial Arveson systems as such, but also their spatial product depends *a priori* on the choice of the reference units of its factors. This immediately raises the question whether different choices of references units yield isomorphic products or not. In [10] this question was answered in the affirmative sense. One aim of the present papers is to show how this universality comes quite naturally from the random set point of view on Arveson systems. Only after knowing the result from a former version of the present paper, [10] achieved the same goal without explicit reference to random sets. Meanwhile, there are follow-up papers [12, 20] which generalise this result.

From (1) it is easy to see that the spatial product is a subsystem of the tensor product system. Nevertheless, the nature of this embedding is not completely clarified. Using the random set construction of [18], we characterise here the embedding of the spatial product into the tensor product easily. These random set structures arise naturally with any embedding $\mathscr{G}\subseteq\mathscr{E}$ of Arveson systems in the following way. Consider the projections

$$\mathrm{P}_{s,t} = \mathbf{1}_{\mathscr{E}_s}\otimes\mathrm{Pr}_{\mathscr{G}_{t-s}}\otimes\mathbf{1}_{\mathscr{E}_{1-t}} \qquad \in\mathscr{B}(\mathscr{E}_1=\mathscr{E}_s\otimes\mathscr{E}_{t-s}\otimes\mathscr{E}_{1-t})$$

on $\mathscr{E}_1$. They fulfil the relation

$$\mathrm{P}_{r,s}\mathrm{P}_{s,t} = \mathrm{P}_{r,t} \qquad 0\leq r\leq s\leq t\leq 1$$

It was one of the results of [18], inspired by [28], to give the interesting part of the (normal and separable) representation theory of these relations, identifying the

projection $\mathrm{P}_{s,t}$ with multiplication by the $\{0,1\}$-valued random variable

$$X_{s,t}(Z) = \begin{cases} 1 & Z \cap [s,t] = \emptyset \\ 0 & Z \cap [s,t] \neq \emptyset \end{cases}$$

on the space $\mathscr{C}_{[0,1]} = \{Z \subseteq [0,1] : Z \text{ closed}\}$ equipped with a suitable probability measure. Multiplicity is encoded in a direct integral of Hilbert spaces as usual, see Theorem 9.3 below. Having the representation for such projections at hand, it is quite easy to compute functions of those projections. In the present situation, we want to compute the projection onto $(\mathscr{E}_u \otimes_v \mathscr{F})_1$ which characterises $\mathscr{E}_u \otimes_v \mathscr{F}$ completely. A few basic facts about the relevant measures then yield independence of the construction from the reference units, solving a question raised by Powers in [21]. This solution was presented also in [10] with a different proof not using the random set structure explicitly. But that proof, unobviously, computed just consequences of the random set structure without reference to it. We hope to convince the reader that using random sets gives a much more clear derivation of the results and that the present paper is worthwhile.

Note that there are examples that the two products form nonisomorphic product systems, provided by [21] together with [2] based on the CP-flow technique. Below, another series of examples is provided. Those examples use the Arveson systems coming from the zero sets of Bessel diffusions as introduced already by TSIRELSON [28]. Those examples are all of type II_0 but nonisomorphic. As a by-product, we show that those product systems are really primitive in the sense that they contain only trivial subsystems. Thus they are also prime product systems in the sense of [15]. Further, spatial products of the Bessel zero Arveson systems have a quite similar structure, with a rich group of automorphisms, compared to the behavior of type I_1 Arveson systems under the (spatial) product. Still, we do not know whether these examples really differ from those in [21]. Still, there does not seems to be a proof that the spatial product is intrinsic using the CP-flow technique.

Acknowledgements This work began with a RiP stay at the Oberwolfach institute in 2007. The main part was completed during a stay at the Università de Molisse, Campobasso, 2009. Special thanks to M. Skeide for the warm hospitality during the latter stay, and for a lot of discussion and encouragement later as well. Further, discussions with B.V.R. Bhat, K. Waldorf, D. Markiewicz and P. Moerters helped a lot to shape this work. Many thanks to an anonymous referee whose suggestions helped to improve the presentation.

2 Continuous product systems of Hilbert spaces

Let us start with some definitions.

Definition 2.1 *An* Arveson system *is a measurable family $\mathscr{E} = (\mathscr{E}_t)_{t\geq 0}$ of separable Hilbert spaces endowed with a measurable family of unitaries $V_{s,t} : \mathscr{E}_s \otimes \mathscr{E}_t \mapsto \mathscr{E}_{s+t}$ for all $s,t \geq 0$ which fulfils for all $r,s,t \geq 0$*

$$V_{r,s+t} \circ (\mathbf{1}_{\mathscr{E}_r} \otimes V_{s,t}) = V_{r+s,t} \circ (V_{r,s} \otimes \mathbf{1}_{\mathscr{E}_t}).$$

Definition 2.2 *A unit u of an Arveson system is a measurable non-zero section $(u_t)_{t\geq 0}$ through $(\mathscr{E}_t)_{t\geq 0}$, which satisfies for all $s,t \geq 0$*

$$u_{s+t} = V_{s,t} u_s \otimes u_t = u_s \otimes u_t.$$

If u is normalised ($\|u_t\| = 1 \forall t \geq 0$), the pair $(\mathscr{E}, u)$ is also called spatial *Arveson system. For any Arveson system $\mathscr{E}$ denote $\mathscr{U}_1(\mathscr{E})$ the set of all normalised units of $\mathscr{E}$.*

Remark 1 *We do not make the definition of measurability more explicit throughout this paper. For a thorough discussion see [18], especially section 7 there. Most importantly, by [18, Theorem 7.7] existence of a compatible measurable structure for an Arveson system is determined by the algebraic structure (given by the family $(V_{s,t})_{0\leq s\leq t}$) alone. The example Arveson systems introduced below obey that condition.*

Another distinction to [3] is the inclusion of the trivial 0- and 1-dimensional product systems and of time 0. This way the order structure of Arveson subsystems becomes simpler.

In the sequel we drop the operators $V_{s,t}$ whenever there is no loss of precision.

Definition 2.3 *Let additionally $\mathscr{F}$ be another Arveson system with unitaries $(W_{s,t})_{0\leq s,t}$.*

1. *We say that $\theta = (\theta_t)_{t\geq 0}$ is an isomorphism of product systems if $\theta_t : \mathscr{E}_t \mapsto \mathscr{F}_t$ is a unitary for all $t \geq 0$ and for all $s,t \geq 0$*

$$\theta_{s+t} \circ V_{s,t} = W_{s,t} \circ (\theta_s \otimes \theta_t).$$

   If $\mathscr{F} = \mathscr{E}$, θ is called automorphism.

2. *We call $\mathscr{F}$ a subsystem of $\mathscr{E}$ if $\mathscr{F}_t \subseteq \mathscr{E}_t$ for all $t \geq 0$ and $W_{s,t} = V_{s,t}|_{\mathscr{F}_s \otimes \mathscr{F}_t}$ for all $s,t \geq 0$.*

Then

$$\mathrm{Aut}(\mathscr{E}) = \{\theta : \theta \text{ is an automorphism of } \mathscr{E}\}$$

is a group under pointwise composition, called gauge group of $\mathscr{E}$.

According to [23], [8] we introduce now another product, the spatial product of Arveson systems. For this and further use later, observe by [18, Theorem 5.7] that for an Arveson system $\mathscr{E}$ the set

$$\mathscr{S}(\mathscr{E}) = \{\mathscr{F} : \mathscr{F} \text{ is an Arveson subsystem of } \mathscr{E}\}$$

forms a (complete) lattice with respect to the fibrewise inclusion order. Thus $\mathscr{E}' \vee \mathscr{F}'$ denotes the smallest Arveson subsystem containing both $\mathscr{E}'$ and $\mathscr{F}'$. Under slight abuse of notation, we identify normalised units u with the subsystem $(\mathbb{C}u_t)_{t\geq 0}$.

Definition 2.4 *Let $(\mathscr{E},u)$ and $(\mathscr{F},v)$ be two spatial Arveson systems. We define their* spatial product *as*

$$\mathscr{E}\,{}_u\otimes_v\mathscr{F} := (u\otimes\mathscr{F}) \vee (\mathscr{E}\otimes v) \subseteq \mathscr{E}\otimes\mathscr{F}$$

For a more explicit definition (see e.g. [8]), let

$$\Pi^t = \{(t_1,\dots,t_n) : n \in \{1,2,\dots\}, t_i > 0, t_1+\cdots+t_n = t\}$$

denote the set of interval partitions of $[0,t]$ (in a suitable parametrisation). We order Π^t by $(t_1,\dots,t_n) \prec (s_1,\dots,s_m)$ if $n \leq m$ and there is a strictly increasing map $\varphi : \{1,\dots,n,n+1\} \mapsto \{1,\dots,m,m+1\}$ with $\varphi(1) = 1$, $\varphi(n+1) = m+1$, and

$$t_i = s_{\varphi(i)} + \cdots + s_{\varphi(i+1)-1} \forall i = 1,\dots,n.$$

Further, for any vector w in a Hilbert space denote $w^\perp$ its orthogonal complement.

Proposition 2.1 ([10, Proposition 2.7]) *Let $(\mathscr{E},u)$ and $(\mathscr{F},v)$ be two spatial Arveson systems. Define Hilbert spaces*

$$G_t^{u,v} = u_t \otimes v_t^\perp \oplus \mathbb{C}u_t \otimes v_t \oplus u_t^\perp \otimes v_t. \tag{2}$$

Then for all $t > 0$

$$(\mathscr{E}\,{}_u\otimes_v\mathscr{F})_t = \lim_{(t_1,\dots,t_n)\in\Pi^t} G_{t_1}^{u,v} \otimes G_{t_2}^{u,v} \otimes \cdots \otimes G_{t_{n-1}}^{u,v} \otimes G_{t_n}^{u,v}. \tag{3}$$

Remark 2 *The work on inclusion systems [9] is a direct generalisation of this inductive limit technique.*

The main question now is whether the inclusion $\mathscr{E}_u \otimes_v \mathscr{F} \subseteq \mathscr{E} \otimes \mathscr{F}$ might be proper. The answer is reported later.

For any (spatial) Arveson system we introduce its type I part

$$\mathscr{E}^{\mathscr{U}} = \bigvee_{u \in \mathscr{U}_1(\mathscr{E})} u,$$

the Arveson subsystem generated by its units. $\mathscr{E}$ is called type I, if $\mathscr{E} = \mathscr{E}^{\mathscr{U}}$, type II if $\mathscr{E}^{\mathscr{U}} \neq 0, \mathscr{E}$, and type III if $\mathscr{E}^{\mathscr{U}} = 0$ or $\mathscr{U}_1(\mathscr{E}) = \emptyset$. $\mathscr{E}^{\mathscr{U}}$ is isomorphic to an Arveson system $(\Gamma(L^2([0,t],\mathscr{K})))_{t\geq 0}$ of symmetric Fock spaces for some separable Hilbert space $\mathscr{K}$ [3]. $\dim \mathscr{K}$ is an invariant called *index of* $\mathscr{E}$. We subclassify the types I, II according to their index. This means, e.g., that for $n \in \mathbb{N}$ an Arveson system of type I_n is isomorphic to $((\Gamma(L^2([0,t],\mathbb{C}^n))))_{t\geq 0}$ [3]. It is easy to see that the index is additive under both the tensor product and the spatial product.

3 Product Systems and Random Sets

If $\mathscr{E}$ is an Arveson system, there is an important unitary one parameter group $(\tau_t)_{t\in\mathbb{R}} \subset \mathscr{B}(\mathscr{E}_1)$ acting for $t \in (0,1)$ with regard to the representations $\mathscr{E}_{1-t} \otimes \mathscr{E}_t \cong \mathscr{E}_1 \cong \mathscr{E}_t \otimes \mathscr{E}_{1-t}$ as *flip*:

$$\tau_t x_{1-t} \otimes x_t = x_t \otimes x_{1-t} \qquad (x_{1-t} \in \mathscr{E}_{1-t},\ x_t \in \mathscr{E}_t). \tag{4}$$

The operators τ_t for $t \notin (0,1)$ are obtained by $1-$periodic continuation. These unitaries yield via

$$\Theta_t(a) = \tau_t^* a \tau_t \qquad (a \in \mathscr{B}(\mathscr{E}_1)),$$

a periodic one parameter automorphism group $(\Theta_t)_{t\in\mathbb{R}}$ on $\mathscr{B}(\mathscr{E}_1)$.

Observe that any Arveson subsystem $\mathscr{G}$ of an Arveson system $\mathscr{E}$ yields a family $(\mathrm{P}^{\mathscr{G}}_{s,t})_{0\leq s<t\leq 1}$ of projections

$$\mathrm{P}^{\mathscr{G}}_{s,t} = \mathbf{1}_{\mathscr{E}_s} \otimes \mathrm{Pr}_{\mathscr{G}_{t-s}} \otimes \mathbf{1}_{\mathscr{E}_{1-t}} \qquad \in \mathscr{B}(\mathscr{E}_1 = \mathscr{E}_s \otimes \mathscr{E}_{t-s} \otimes \mathscr{E}_{1-t}). \tag{5}$$

This family fulfils the following relations

$$\begin{aligned} \mathrm{P}^{\mathscr{G}}_{s,t}\mathrm{P}^{\mathscr{G}}_{t,u} &= \mathrm{P}^{\mathscr{G}}_{s,u} \qquad 0 \leq s \leq t \leq u \leq 1 \\ \mathrm{P}^{\mathscr{G}}_{s+u,t+u} &= \Theta_u(\mathrm{P}^{\mathscr{G}}_{s,t}) \qquad 0 \leq s \leq t \leq 1, -s \leq u \leq 1-t. \end{aligned}$$

The following theorem makes the rôle of (distributions of) random sets in Arveson systems apparent. Thereby, let $\mathscr{C}_{[0,1]}$ denote the space of closed subsets of the unit interval. It is a compact metric space itself, with a corresponding σ-field of Borel sets. We implicitly assume all probability measures on $\mathscr{C}_{[0,1]}$ to be defined on this σ-field.

Theorem 3.1 ([18, Theorem 3.16]) *Let $\mathscr{E}$ be an Arveson system, ω be a faithful normal state on $\mathscr{B}(\mathscr{E}_1)$ and $\mathscr{G}$ be an Arveson subsystem of $\mathscr{E}$.*

Then there is a unique probability measure μ_ω on $\mathscr{C}_{[0,1]}$ with

$$\mu_\omega(\{Z : Z \cap (\bigcup_i [s_i,t_i]) = \emptyset\}) = \omega(\mathrm{P}^{\mathscr{G}}_{s_1,t_1} \cdots \mathrm{P}^{\mathscr{G}}_{s_k,t_k}) \qquad (0 \le s_i < t_i \le 1)$$

Further, there is a unique normal isomorphism $j_{\mathscr{G}}$,

$$j_{\mathscr{G}} : L^\infty(\mu_\omega) \mapsto \{\mathrm{P}^{\mathscr{G}}_{s,t} : 0 \le s < t \le 1\}'' \subset \mathscr{B}(\mathscr{E}_1),$$

with

$$j_{\mathscr{G}}(1_{\{Z\cap[s,t]=\emptyset\}}) = \mathrm{P}^{\mathscr{G}}_{s,t} \qquad (0 \le s < t \le 1).$$

4 Stationary factorising measure types

We saw above that the space $L^\infty(\mu_\omega)$ seems to play a more fundamental rôle than the measure μ_ω itself. That means, equivalent measures yield the same structure. We want to formalise this.

Recall that a *measure type* is an equivalence class of probability measures, where equivalence of measures μ and ν (symbol $\mu \sim \nu$) means that μ and ν have the same null sets.

On $\mathscr{C}_{[0,1]}$, we have the natural operations of restriction $Z \mapsto Z_{s,t} = Z \cap [s,t]$ and circular shift $Z \mapsto Z+t := \overline{Z+t \pmod 1}$. The first gives rise to an image measure $\mu_{s,t}$, the second to the image measure $\mu + t$. The convolution associated with $\cup$ is denoted by $*$. These notions transfer naturally to measure types.

Definition 4.1 *A measure type $\mathscr{M}$ on $\mathscr{C}_{[0,1]}$ is* stationary factorising *if*

$$\begin{aligned} \mathscr{M}_{r,t} &= \mathscr{M}_{r,s} * \mathscr{M}_{s,t} && (0 \le r < s < t \le 1) \\ \mathscr{M}_{r,s} + t &= \mathscr{M}_{r+t,s+t} && (0 \le r < s < s+t \le 1) \end{aligned}$$

Theorem 4.1 ([18, Theorem 3.22 and Corollary 6.2]) *In the situation of Theorem 3.1,*

$$\mathscr{M}^{\mathscr{G}} = \{\mu_\omega : \omega \textit{ faithful }\}$$

is a stationary factorising measure type.

5 The embedding $\mathscr{E}_u \otimes_v \mathscr{F} \subseteq \mathscr{E} \otimes \mathscr{F}$

We use also the following extension of Theorem 3.1:

Proposition 5.1 ([18, Proposition 3.32]) *Suppose for two subsystems $\mathscr{G}_1,\mathscr{G}_2$ of an Arveson system that the projection families $\mathrm{P}^{\mathscr{G}_1},\mathrm{P}^{\mathscr{G}_2}$ commute.*

Then there exists for all normal states ω on $\mathscr{B}(\mathscr{E}_1)$ a unique probability measure μ_ω on $\mathscr{C}_{[0,1]}\times\mathscr{C}_{[0,1]}$ with

$$\mu_\omega(\Big\{(Z_1,Z_2): Z_j\cap\bigcup_i[s_i^j,t_i^j]=\emptyset\Big\}) = \omega(\prod_j\prod_i \mathrm{P}^{\mathscr{G}_j}_{s_i^j,t_i^j}).$$

The corresponding measure type is denoted $\mathscr{M}^{\mathscr{G}_1,\mathscr{G}_2}$.

Further, there exists unique isomorphism $J_{\mathscr{G}_1,\mathscr{G}_2}: L^\infty(\mathscr{M}^{\mathscr{G}_1,\mathscr{G}_2})\mapsto\mathscr{B}(\mathscr{E}_1)$ with

$$J_{\mathscr{G}_1,\mathscr{G}_2}(1_{\{(Z_1,Z_2):Z_j\cap[s,t]=\emptyset\}}) = P^j_{s,t} \qquad (j=1,2).$$

Denote for a closed set $Z\subseteq\mathbb{R}_{\geq 0}$ the set of its limit points by $\hat{Z}$. I.e.,

$$\hat{Z} = \Big\{t\in Z: t\in\overline{Z\setminus\{t\}}\Big\} = \Big\{t\in Z: \exists Z\ni t_n\neq t, n\in\mathbb{N}, t=\lim_{n\to\infty}t_n\Big\}.$$

This means that $Z\setminus\hat{Z}$ is the countable set of isolated points of Z.

Example 5.1 ([18, Proposition 3.33]) *Consider $\mathscr{G}_1=\mathbb{C}u$ for a unit $(u_t)_{t\geq 0}$ and $\mathscr{G}_2=\mathscr{E}^{\mathscr{U}}$. Then*

$$J_{\mathscr{G}_1,\mathscr{G}_2}(f) = J_{u,\mathscr{E}^{\mathscr{U}}}(f) = J_u(g)$$

where $g(Z)=f(Z,\hat{Z})$.

Proposition 5.2 *For spatial Arveson systems $(\mathscr{E},u)$, $(\mathscr{F},v)$ it holds*

$$\mathrm{P}^{\mathscr{E}_u\otimes_v\mathscr{F}}_{s,t} = J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}(1_{\{(Z_1,Z_2):Z_1\cap Z_2\cap[s,t]=\emptyset\}}) \tag{6}$$

Remark 3 *Compare this expression to [21, Theorem 2.1], which seems to compute $\mathrm{P}^{\mathscr{G}}_{0,1}$ in a special case. Observe that the latter projection identifies already the corresponding Arveson subsystem.*

Proof. We use Proposition 2.1. Using the notation (2) we derive

$$G^{u,v}_t\otimes\mathscr{E}_{1-t}\otimes\mathscr{F}_{1-t} = J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}(1_{\{(Z_1,Z_2):Z_1\cap[0,t]=\emptyset \text{ or } Z_2\cap[0,t]=\emptyset\}}).$$

By normality of $J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}$ we obtain

$$\begin{aligned}
\mathrm{P}^{\mathscr{E}_u\otimes_v\mathscr{F}}_{0,1} &= \lim_{(t_1,\ldots,t_n)\in\Pi^1}\mathrm{Pr}_{G^{u,v}_{t_1}}\otimes\mathrm{Pr}_{G^{u,v}_{t_2-t_1}}\otimes\cdots\otimes\mathrm{Pr}_{G^{u,v}_{t_n-t_{n-1}}}\\
&= \lim_{(t_1,\ldots,t_n)\in\Pi^1} J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}(1_{\Big\{(Z_1,Z_2):\forall i: Z_1\cap[\sum_{j=1}^i t_j,\sum_{j=1}^{i+1}t_j]=\emptyset \text{ or } Z_2\cap[\sum_{j=1}^i t_j,\sum_{j=1}^{i+1}t_j]=\emptyset\Big\}})\\
&= J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}(\lim_{(t_1,\ldots,t_n)\in\Pi^1} 1_{\Big\{(Z_1,Z_2):\forall i: Z_1\cap[\sum_{j=1}^i t_j,\sum_{j=1}^{i+1}t_j]=\emptyset \text{ or } Z_2\cap[\sum_{j=1}^i t_j,\sum_{j=1}^{i+1}t_j]=\emptyset\Big\}})\\
&= J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}(1_{\{(Z_1,Z_2):Z_1\cap Z_2=\emptyset\}}).
\end{aligned}$$

Formula (6) for $s \neq 0$ or $t \neq 1$ follows immediately since $\mathrm{P}_{0,1}^{\mathscr{E}_u \otimes_v \mathscr{F}}$ determines the whole Arveson system $\mathscr{E}_u \otimes_v \mathscr{F}$. This completes the proof. ∎

Proposition 5.3 *The relation $\mathscr{E}_u \otimes_v \mathscr{F} = \mathscr{E} \otimes \mathscr{F}$ is valid if and only if*

$$Z_1 \cap Z_2 = \emptyset \qquad (\mathscr{M}^u \otimes \mathscr{M}^v - a.s.) \tag{7}$$

if and only if

$$\widehat{Z_1} \cap \widehat{Z_2} = \emptyset \qquad (\mathscr{M}^u \otimes \mathscr{M}^v - a.s.) \tag{8}$$

Proof. The first assertion is clear. The second one follows from the fact that $Z_1 \setminus \widehat{Z_1}$ and $Z_2 \setminus \widehat{Z_2}$ are countable. Since $(\mathscr{E}, u)$ and $(\mathscr{F}, v)$ are spatial, both $Z_1$and Z_2 are different from $[0,1]$ almost surely. Then we know from [18, Proposition 4.4] that such a stationary factorising random set almost never meets a countable set and we conclude

$$Z_1 \cap Z_2 = \widehat{Z_1} \cap Z_2 = \widehat{Z_1} \cap \widehat{Z_2} \qquad (\mathscr{M}^u \otimes \mathscr{M}^v - \text{a.s.})$$

This completes the proof. ∎

Corollary 5.1 *If the lattice $\mathscr{S}(\mathscr{E}_u \otimes_v \mathscr{F})$ has finite depth and* (7) *is not fulfilled, $\mathscr{E} \otimes \mathscr{F} \ncong \mathscr{E}_u \otimes_v \mathscr{F}$.*

Proof. If (7) is not valid, $\mathscr{E}_u \otimes_v \mathscr{F}$ is a proper subsystem of $\mathscr{E} \otimes \mathscr{F}$. If both were isomorphic, iteration of this observation would yield an infinite chain of Arveson subsystems in $\mathscr{S}(\mathscr{E}_u \otimes_v \mathscr{F})$. ∎

Corollary 5.2 *In the following cases we have that $\mathscr{E}_u \otimes_v \mathscr{F} = \mathscr{E} \otimes \mathscr{F}$:*

1. *One of $\mathscr{E}$ or $\mathscr{F}$ is type* I.
2. *Z is countable $\mathscr{M}^u$-a.s. or $\mathscr{M}^v$-a.s.*

Proof. 1. Suppose $\mathscr{F}$ is type I. Then $\widehat{Z} = \emptyset$ $\mathscr{M}^v$-a.s., since Z is $\mathscr{M}^v$-a.s. finite by [18, Proposition 3.33]. (8) gives the desired conclusion.

2. [18, Proposition 4.4] yields again the conclusion.

∎

6 The spatial product does not depend on the units

A direct consequence of Proposition 5.3 is that $\mathscr{E}_u\otimes_v\mathscr{F}$ is intrinsic, i.e. it does not depend on the choice of u and v. Another formulation of the proof without explicit reference to random sets can be found in [10, Theorem 3.1].

Theorem 6.1 *Let $(\mathscr{E},u)$, $(\mathscr{E},u')$, $(\mathscr{F},v)$ and $(\mathscr{F},v')$ be spatial Arveson systems. Then*

$$\mathscr{E}_u\otimes_v\mathscr{F} = \mathscr{E}_{u'}\otimes_{v'}\mathscr{F}.$$

Proof. We know from [18, Proposition 3.33] for $f\in L^\infty(\mathscr{M}^u)$ that $J_u(f\circ\widehat{\cdot}) = J_{\mathscr{E}^{\mathscr{U}}}(f)$. By Proposition 5.3 this shows

$$\begin{aligned}\mathrm{P}^{\mathscr{E}_u\otimes_v\mathscr{F}}_{s,t} &= J_{\mathscr{E}\otimes v,u\otimes\mathscr{F}}\big(1_{\{(Z_1,Z_2):\widehat{Z_1}\cap\widehat{Z_2}\cap[s,t]=\emptyset\}}\big)\\ &= J_{\mathscr{E}\otimes\mathscr{F}^{\mathscr{U}},\mathscr{E}^{\mathscr{U}}\otimes\mathscr{F}}\big(1_{\{(Z_1,Z_2):Z_1\cap Z_2\cap[s,t]=\emptyset\}}\big).\end{aligned}$$

The last expression is independent of u and v. ∎

Corollary 6.1 *It holds*

$$\mathscr{E}_u\otimes_v\mathscr{F} = (\mathscr{E}^{\mathscr{U}}\otimes\mathscr{F})\vee(\mathscr{E}\otimes\mathscr{F}^{\mathscr{U}}).$$

Thus, we use $\mathscr{E}\otimes_{\mathscr{U}}\mathscr{F}$ as new symbol for $\mathscr{E}_u\otimes_v\mathscr{F}$. This is also consistent with the amalgamation procedure from [9]. Note that [10] introduced the symbol $\mathscr{E}\otimes^0\mathscr{F}$.

7 From measure types to Hilbert spaces

Before we study special examples of Arveson systems, we want to present the general mechanism for constructing those examples. It dates back to TSIRELSON [28].

If $\mu\sim\mu'$ are two measures on the same space (here $\mathscr{C}_{[0,1]}$), the abelian von Neumann algebras $L^\infty(\mu)$ and $L^\infty(\mu')$ coincide, and we observe a canonical space $L^\infty(\mathscr{M})$ if $\mathscr{M}$ is the measure type of μ and μ'. Now we want to present an intrinsic construction of a Hilbert space $L^2(\mathscr{M})$. In this we follow [28, 29] or originally [1].

Define for any $\mu,\mu'\in\mathscr{M}$ a unitary $U_{\mu,\mu'}: L^2(\mu)\mapsto L^2(\mu')$ through

$$U_{\mu,\mu'}\psi(Z) = \sqrt{\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}(Z)}\,\psi(Z) \qquad (\psi\in L^2(\mu),\mu-\text{a.a. } Z\in\mathscr{C}_{[0,1]}). \tag{9}$$

Then

$$L^2(\mathscr{M}) = \left\{(\psi_\mu)_{\mu\in\mathscr{M}} : \psi_\mu \in L^2(\mu)\forall\mu\in\mathscr{M}, \psi_{\mu'} = U_{\mu,\mu'}\psi_\mu\forall\mu,\mu'\in\mathscr{M}\right\} \quad (10)$$

is a Hilbert space with the inner product

$$\langle\psi,\psi'\rangle_{L^2(\mathscr{M})} = \int \overline{\psi_\mu}\psi'_\mu \mathrm{d}\mu.$$

This inner product is independent from the choice of $\mu\in\mathscr{M}$.

Now we obtain

Proposition 7.1 ([18, Proposition 4.3]) *Let $\mathscr{M}$ be a stationary factorising measure type on $\mathscr{C}_{[0,1]}$ different from $\left\{\delta_{[0,1]}\right\}$. Define operators $V_{s,t} : L^2(\mathscr{M}_{0,s})\otimes L^2(\mathscr{M}_{0,t}) \mapsto L^2(\mathscr{M}_{0,s+t})$ for $0\le s,t$, $s+t\le 1$ through*

$$(V_{s,t}\psi\otimes\psi')_{\mu_s*(\mu'_t+s)}(Z) = \psi_{\mu_s}(Z\cap[0,s])\psi'_{\mu'_t}(Z\cap[s,s+t]-s)$$

Then $V_{s,t}$ are well-defined unitaries and give rise to an Arveson system $\mathscr{E} = \mathscr{E}^{\mathscr{M}} = (\mathscr{E}_t)_{t\ge0}$ with $\mathscr{E}_t = L^2(\mathscr{M}_{0,t})$ for $0\le t\le 1$.

One unit $u\in\mathscr{U}_1(\mathscr{E})$ is determined by

$$(u_t)_{\mu_{0,t}}(Z) = \mu_{0,t}(\{\emptyset\})^{-1/2}\,1_{\{\emptyset\}}(Z)$$

for $t\in[0,1]$. Then $\mathscr{M}^u = \mathscr{M}$.

All examples of such measure types used in this paper come from hitting sets of strong Markov processes $(X_t)_{t\ge0}$. Basically, such sets are constructed by

$$Z = \{t+\tau : X_t = x^*\}$$

where x^* is a suitable point and τ is a random variable independent from $(X_t)_{t\ge0}$ with law equivalent to Lebesgue measure on $\mathbb{R}_{\ge0}$. Please note that only almost sure properties of these random sets are important, not the special probabilistic structure. E.g., without loss of generality, we may assume $\tau\sim\mathrm{Exp}_1$.

If x^* is a suitable point then there is a nonnegative right-continuous increasing process $(M_s)_{s\ge0}$ with stationary independent increments upto a certain life time such that conditional on $X_0 = x^*$,

$$\{t : X_t = x^*\} = \overline{\{M_s : s\ge0\}}.$$

$(M_s)_{s\ge0}$ is called *subordinator*, see [5] for a thorough account on these processes and their range.

For a coarse classification of those random sets, remember the definition of Hausdorff-dimension of a set Z. For $\alpha > 0$ the α-dimensional *Hausdorff measure* of a Borel set Z is defined as

$$H^\alpha(Z) = \sup_{\varepsilon>0} H^\alpha_\varepsilon(Z), \tag{11}$$

where

$$H^\alpha_\varepsilon(Z) = \inf\left\{ \sum_{i\in\mathbb{N}} \Delta(B_i)^\alpha : (B_i)_{i\in\mathbb{N}} \text{ are sets with } \Delta(B_i) \le \varepsilon \text{ and } \bigcup_{i\in\mathbb{N}} B_i \supseteq Z \right\}, \tag{12}$$

denoting $\Delta(B)$ the diameter of B. Then the *Hausdorff dimension* $\dim_H Z$ of a Borel set Z is defined by

$$\dim_H Z = \inf\{\alpha > 0 : H^\alpha(Z) > 0\}.$$

We consider even more special hitting sets, coming from Bessel diffusions:

Example 7.1 ([28]) *Let $\left(X_t^{(d)}\right)_{t\ge 0}$ be a Bessel diffusion with parameter $d > 0$ starting in a point $x_0 > 0$. This means $\left(X_t^{(d)}\right)_{t\ge 0}$ is a strong Markov (diffusion) process on $\mathbb{R}_{\ge 0}$ with generator*

$$\frac{\mathrm{d}\mathbb{E}_x f(X_t^{(d)})}{\mathrm{d}t}\Big|_{t=0} = \frac{1}{2} f''(x) + \frac{d-1}{2x} f'(x).$$

Throughout this work, let $\mathbb{E}_x$ and $\mathbb{P}_x$ denote the conditional expectation and conditional probability given $X_0 = x$ respectively. For $d \in \mathbb{N}$ we could realise this process via $X_t^{(d)} = \left\|B_t^d\right\|$, where $\left(B_t^d\right)_{t\ge 0}$ is d-dimensional Brownian motion. In the general case, the Bessel process is also defined as the (unique) nonnegative solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t + \frac{d-1}{2}\frac{1}{X_t}\mathrm{d}t.$$

Then we write $\left(X_t^{(d)}\right)_{t\ge 0} \sim \mathrm{BES}(d, x_0)$.

According to the above mentioned scheme, define a random closed set $Z \in \mathscr{C}_{[0,1]}$ by

$$Z = \left\{ t \ge 0 : X_t^{(d)} = 0 \right\} \cap [0,1]$$

Observe that in this case the subordinator is stable of index d [5]. This means

$$\mathbb{E}\mathrm{e}^{-\lambda M_s} = \mathrm{e}^{s\lambda^d}.$$

Moreover, for $d \geq 2$ $Z = \emptyset$ a.s. So we restrict to $d \in (0,2)$ for the rest of the paper.

Then the measure type $\mathscr{M}_d = \{\mu : \mu \sim \mathscr{L}(Z)\}$, which does not depend on x_0, is stationary factorising. Moreover, $\mathscr{M}_d$-a.s. the set Z has Hausdorff dimension $1 - \frac{d}{2}$ near every of its points. This means for all (s,t) with $Z \cap (s,t) \neq \emptyset$ it holds $\dim_H(Z \cap (s,t)) = 1 - \frac{d}{2}$.

As a consequence Z has no isolated points: $\widehat{Z} = Z$. This immediately implies that the Arveson system $(\mathscr{E}_t)_{t \geq 0}$ determined by $\mathscr{E}_t = L^2(\mathscr{M}_{0,t})$, $t \in [0,1]$, is type II_0 *[18, Corollary 4.7], [28]. In the sequel, we denote this Arveson system by $\mathscr{E}^d$. Further, $\widehat{Z} = Z$ $\mathscr{M}_d-$a.s. also implies $\mathscr{M}_d = \mathscr{M}^{\mathscr{U}}$. The latter measure type is an invariant of $\mathscr{E}$ by [18, Theorem 3.22] and we conclude that $\mathscr{E}^d \ncong \mathscr{E}^{d'}$ for $d' \neq d$ (as long as both are < 2), see also [28].*

One more construction is useful in the sequel: The local time of the diffusion in 0. This local time, denoted $(L_t)_{t \geq 0}$, is the inverse of the subordinator $(M_s)_{s \geq 0}$:

$$L_t = \sup\{s > 0 : \tau + M_s \leq t\}$$

Since $t \mapsto L_t$ is a random increasing nonnegative function, it is the cumulative distribution function of a random measure. It is easy to see that the support of this measure is just Z. By results of [13] this measure is just the restriction of a certain Hausdorff measure to Z. Thus this random measure depends on Z only and we write $L_t(Z)$.

8 Bessel zeros yields primitive Arveson systems

Definition 8.1 *A spatial Arveson system $(\mathscr{E}, u)$ is called primitive, if $\mathscr{S}(\mathscr{E}) = \{0, u, \mathscr{E}\}$.*

A spatial Arveson system $(\mathscr{E}, u)$ is prime (spatially prime), if for Arveson systems $\mathscr{F}, \mathscr{G}$ with $\mathscr{F} \otimes \mathscr{G} = \mathscr{E}$ ($\mathscr{F} \otimes_{\mathscr{U}} \mathscr{G} = \mathscr{E}$) it follows that either $\mathscr{F}$ or $\mathscr{G}$ is trivial, i.e. it is isomorphic to $(\mathbb{C})_{t \geq 0}$.

According to [18, Proposition 4.32, Note 4.33] for all $k = 1, 2 \ldots$ there are uncountably many examples of type II_k Arveson systems which are prime and spatially prime. We now focus on examples of prime type II_0 Arveson systems.

In [15] there was derived a useful criterion for Arveson systems to be prime:

Proposition 8.1 *If for a spatial Arveson system $(\mathscr{E}, u)$ the lattice $\mathscr{S}(\mathscr{E})$ is totally ordered then $\mathscr{E}$ is both prime and spatially prime.*

Especially, primitive Arveson systems are both prime and spatially prime.

Proof. Analogous to [15]. ∎

The aim of the present section is the proof of

Theorem 8.1 *$\mathscr{E}^d$ is primitive for all $0 < d < 2$.*

Remark 4 *This solves a question raised in [18, Example 5.16]. To our knowledge, these are the first proven nontrivial examples of primitive Arveson systems.*

For a proof, we still need some more structure.

Definition 8.2 *Suppose $\mathscr{M}$ is a stationary factorising measure type on $\mathscr{C}_{[0,1]}$. Then an $\mathscr{M}$-local stationary opening is a measurable map $\varphi : \mathscr{C}_{[0,1]} \mapsto \mathscr{C}_{[0,1]}$ with*

(i) $\varphi(Z) \subseteq Z$ *for all* $Z \in \mathscr{C}_{[0,1]}$,

(ii) $\varphi(Z+t) = \varphi(Z)+t$ *for all* $t \geq 0$ *and $\mathscr{M}$-a.a. Z, and*

(iii)

$$\varphi(Z \cap [s,t]) = \varphi(Z) \cap [s,t]$$

for all $0 < s < t \leq 1$ *for $\mathscr{M}$-a.a. Z.*

Remark 5 *The name "opening" for operators with property (i) is common in mathematical morphology, see e.g. [14].*

The importance of this notion lies in

Proposition 8.2 ([18, Lemma 5.14]) *Let $\mathscr{M}$ be a stationary factorising measure type on $\mathscr{C}_{[0,1]}$ and $\mathscr{E} = \mathscr{E}^{\mathscr{M}}$ the associated Arveson system. Suppose $\mathscr{E}$ is type* II_0 *and $\mathscr{F} \neq 0$ is a subsystem of $\mathscr{E}$.*

Then there exists an $\mathscr{M}$-local stationary opening φ such that

$$\mathrm{P}^{\mathscr{F}}_{s,t} = 1_{\{Z:\varphi(Z)\cap[s,t]=\emptyset\}} \qquad (0 < s < t \leq 1).$$

Conversely, every $\mathscr{M}$-local stationary opening gives rise to a nonzero Arveson subsystem this way.

The next proposition is concerned with the probabilistic characterisation of Arveson subsystems of $\mathscr{E}^d$, or more generally Arveson systems arising from measure types of hitting sets of strong Markov processes. For a stochastic process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \Sigma, \mathbb{P})$ introduce the canonical (augmented) filtration Σ^X,

$$\Sigma^X_t = \bigcap_{\varepsilon > 0} \sigma(\{X_s : s \leq t+\varepsilon\} \cup \{B \in \Sigma : \mathbb{P}(B) = 0\})$$

Proposition 8.3 *Let $(X_t)_{t\geq 0}$ be a strong Markov process with a.s. continuous paths in $\mathbb{R}^m$ such that for some $x^* \in \mathbb{R}^m$ the distribution of*

$$Z^X = \{t \in [0,1] : X_t = x^*\} \in \mathscr{C}_{[0,1]}$$

is quasistationary and quasifactorising with measure type $\mathscr{M}$.

If the filtration Σ^X is right continuous then any $\mathscr{M}$-local stationary opening fulfils either $\varphi(Z) = \emptyset$ $\mathbb{P}$-a.s. or $\varphi(Z) = Z$ $\mathbb{P}$-a.s.

Proof. For realisations with $Z^X = \emptyset$ there is nothing to prove. We introduce the random variable $\tau = \inf\{t > 0 : X_t = x^*\}$ such that $X_\tau = x^*$. Then the random variable

$$Y = \begin{cases} 1 & \tau \in \varphi(Z^X) \\ 0 & \tau \notin \varphi(Z^X) \end{cases}$$

is well-defined. Now by the strong Markov property, the process $\left(\tilde{X}_t\right)_{t\geq 0}$, $\tilde{X}_t = X_{\tau+t}$, is distributed according to $\mathbb{P}_{x^*}$. By definition and locality of φ, Y is $\bigcap_{\varepsilon>0}\Sigma^{\tilde{X}}_\varepsilon$-measurable. Thus, by the Blumenthal 0-1 law, $\mathbb{P}(Y = 1) \in \{0,1\}$. Moreover, from [17, Theorem 22.13] we know that

$$Z^X = \overline{\{M_s + \tau : s \geq 0\}} \cap [0,1]$$

where $(M_s)_{s\geq 0}$ is the subordinator associated with X and x^* which is independent of τ. It follows from the time symmetry of subordinators that we can apply the same arguments to the set $T - Z^X_{0,T}$. This means for $\tau_T = \sup\{t < T : X_t = x^*\}$ that $\mathbb{P}(\tau_T \in \varphi(Z^X)) \in \{0,1\}$, too. Introduce for all $q \in \mathbb{Q} \cap \mathbb{R}_{\geq 0}$ random variables $Y^\pm_q \in \{0,1\}$:

$$Y^+_q = \begin{cases} 1 & \text{if} \quad \inf(Z^X \cap (q,\infty)) = \inf(\varphi(Z^X) \cap (q,\infty)) \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y^-_q = \begin{cases} 1 & \text{if} \quad \sup(Z^X \cap (0,q)) = \sup(\varphi(Z^X) \cap (0,q)) \\ 0 & \text{otherwise} \end{cases} .$$

It is easy to see from quasistationarity and quasifactorisation that there exists a fixed $y \in \{(0,0),(0,1),(1,0),(1,1)\}$ such that it holds $\mathbb{P}$-a.s. $(Y^-_q, Y^+_q) = y$ for all positive $q \in \mathbb{Q} \cap \mathbb{R}_{\geq 0}$.

It is clear that $y = (0,0)$ implies $\varphi(Z) = \emptyset$. Similarly, $y = (1,1)$ implies $\varphi(Z) = Z$.

Let us exclude $y = (0,1)$. Choose some $t \in Z^X \setminus \varphi(Z^X)$ and $q_n \nearrow_{n\to\infty} t$, $q_n \in \mathbb{Q} \cap (0,t)$. Then $Y^+_{q_n} = 1$ indicates that there are $t_n \in \varphi(Z^X)$, $q_n < t_n < t$. This implies $\lim_{n\to\infty} t_n = t$. Since $\varphi(Z^X)$ is closed, $t \in \varphi(Z^X)$ contradicting $t \in Z^X \setminus \varphi(Z^X)$.

The case $y = (1,0)$ is excluded by the same arguments. This completes the proof. ∎

Remark 6 *There is the more general bar code construction of [29] giving a vast resource for examples of quasistationary quasifactorising random sets from hitting times sets of diffusions. Unfortunately, Proposition 8.3 does not apply in general, for the hitted set used in [29] is not point like.*

Proof of Theorem 8.1. The claim follows from application of the previous result and Proposition 8.2 to $x^* = 0$ and $(X_t)_{t\geq 0} \sim \mathrm{BES}(d, x_0)$. ∎

We can prove even more than primitivity for $\mathscr{E}^d$, its gauge group is two-dimensional. Remember the definition of $L_t(Z)$ from Example 7.1

Theorem 8.2 *Any $\theta \in \mathrm{Aut}(\mathscr{E}^d)$ has the form*

$$\theta_t f(Z) = \mathrm{e}^{\mathrm{i}(\gamma_0 t + \gamma_1 L_t(Z))} f(Z)$$

for some real γ_0, γ_1.

Remark 7 *A similar theorem holds for endomorphisms.*

Proof. We know that θ should leave $\mathscr{E}^{\mathscr{U}}$ invariant. Thus there is $\gamma_0 \in \mathbb{R}$ such that

$$\theta_t u_t = \mathrm{e}^{\mathrm{i}\gamma_0 t} u_t$$

for the standard unit of $\mathscr{E}^d$. Without loss of generality, let $\gamma_0 = 0$. Then θ_1 shall commute with all the projections $\mathrm{P}^u_{s,t}$ defined by the unit through (5). But those projections generate a maximal abelian von Neumann subalgebra of $\mathscr{B}(\mathscr{E}_1)$. Thus θ_1 is in this subalgebra and we find a measurable function $\lambda : \mathscr{C}_{[0,1]} \mapsto \mathbb{T}$ such that $\theta_1 f(Z) = \lambda(Z) f(Z)$. Now we obtain from $\gamma_0 = 0$ that $\lambda(\emptyset) = 1$. Furthermore, for all t and $\mathscr{M}_d-$a.a. $Z \in \mathscr{C}_{[0,1]}$ it must hold

$$\begin{aligned} \lambda(Z+t) &= \lambda(Z) \\ \lambda(Z) &= \lambda(Z_{0,t})\lambda(Z_{t,1}). \end{aligned}$$

From these relations, we could extend λ to $\bigcup_{n\geq 1} \mathscr{C}_{[0,n]}$, e.g. on $\mathscr{C}_{[0,2]}$

$$\lambda(Z) = \lambda(Z_{0,1})\lambda(Z_{1,2} - 1).$$

Suppose now $Z \in \mathscr{C}_{\mathbb{R}_{\geq 0}}$ is the full zero set of a Bessel process with first hitting time τ.

Remember the definition of the subordinator $(M_s)_{s\geq 0}$ from Example 7.1. Then it is easy to see from the strong Markov property and measurability of λ that the S^1-valued process $(\eta_s)_{s\geq 0}$,

$$\eta_s(Z) = \lambda(Z \cap [0, \tau + M_s])$$

has stationary independent multiplicative increments and measurable paths. Fix $\varepsilon > 0$. Then the latter property shows that the set $S^\varepsilon(Z)$ of times s, where η makes larger jumps than ε, is locally finite almost surely. Consequently,

$$\phi(Z) = \{\tau + M_s : s \in S^\varepsilon(Z)\}$$

is an $\mathscr{M}_d$-local stationary opening with $\phi(Z) \subsetneq Z$. By Proposition 8.2, $\phi(Z) = \emptyset$ a.s. Since ε was arbitrary, η must have continuous paths a.s.

As a consequence, there is almost surely a continuous version of

$$t \mapsto \frac{1}{\mathrm{i}} \log \lambda(Z \cap [0,t]) = \zeta_t(Z).$$

Clearly, $(\zeta_t)_{t\geq 0}$ is an additive functional of the Bessel process. Since $\lambda(\emptyset) = 1$ ζ changes only on the zero set Z. By [17, Theorem 19.24], ζ has to be a multiple of the local time. Thus there is some $\gamma_1 \in \mathbb{R}$ such that

$$\lambda(Z) = \mathrm{e}^{\mathrm{i}\gamma_1 L_1(Z)}$$

for $\mathscr{M}_d-$a.a. $Z \in \mathscr{C}_{[0,1]}$. As θ_1 determines θ, this completes the proof. ∎

9 Products of Arveson systems of Bessel zeros

Now we want to analyse the spatial and tensor products of the Arveson systems $\mathscr{E}^d, \mathscr{E}^{d'}$.

First we want to check the condition from Proposition 5.3. Remember that $\dim_H(Z)$ is the Hausdorff dimension of any set $Z \subseteq [0,1]$.

Theorem 9.1 *Assume that $0 < d_1, d_2 < 2$ and let the distribution of Z_i, $i = 1,2$ be from $\mathscr{M}_{d_i}$.*

If $d_1 + d_2 \geq 2$ then almost surely $Z_1 \cap Z_2 = \emptyset$ and $\mathscr{E}^{d_1} \otimes_{\mathscr{U}} \mathscr{E}^{d_2} = \mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}$.

If $d_1 + d_2 < 2$ then with positive probability $Z_1 \cap Z_2 \neq \emptyset$. Furthermore, almost surely for all $s < t$ with $Z_1 \cap Z_2 \cap (s,t) \neq \emptyset$

$$\dim_H(Z_1 \cap Z_2 \cap (s,t)) = 1 - \frac{d_1 + d_2}{2} \tag{13}$$

Consequently, in this case,

$$\mathscr{E}^{d_1} \otimes_{\mathscr{U}} \mathscr{E}^{d_2} \subsetneq \mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}.$$

Proof. By a result of SHIGA AND WATANABE [26], we know that for Bessel processes $(X_t)_{t\geq 0} \sim \mathrm{BES}(d,x)$, $(X'_t)_{t\geq 0} \sim \mathrm{BES}(d',x')$ the process Y,

$$Y_t = \sqrt{X_t^2 + (X'_t)^2}$$

is distributed as $(Y_t)_{t\geq 0} \sim \mathrm{BES}(d+d', \sqrt{x^2+(x'^2)})$. Now

$$\{t : X_t = 0, X_t' = 0\} = \{t : Y_t = 0\}$$

Since we know that for any $0 \leq s < t$ $\mathscr{M}_{d+d'}$-a.s. either $Z \cap [s,t] = \emptyset$ or $\dim_H(Z \cap [s,t]) = 1 - \frac{d+d'}{2}$ this proves the required statements.

Proposition 5.3 completes the proof. ∎

Remark 8 *Please note that we used the special structure of $\mathscr{M}_d$ here. Nevertheless, most of the implications hold true in much more generality, using techniques from [16] to compute Hausdorff dimensions of stationary random sets.*

Theorem 9.2 *Suppose $d_1 \neq d_2$, $0 < d_1, d_2 < 2$.*

Then

$$\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2} \cong \mathscr{E}^{\mathscr{M}_{d_1} * \mathscr{M}_{d_2}}.$$

Moreover, $\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}$ has at most 5 proper subsystems: 0, $u \otimes v$, $u \otimes \mathscr{E}^{d_2}$, $\mathscr{E}^{d_1} \otimes v$, and $\mathscr{E}^{d_1} \otimes_{\mathscr{U}} \mathscr{E}^{d_2}$. The last one appears if and only if $d_1 + d_2 < 2$. Then it is not isomorphic to $\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}$.

Proof. Since the index of Arveson systems is additive, $\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}$ is of type II_0 again. Thus it has only one one-dimensional subsystem $u \otimes v$. The measure type related to this embedding comes from the distribution of $Z_1 \cup Z_2$ where (Z_1, Z_2) has a distibution from $\mathscr{M}_{d_1} \otimes \mathscr{M}_{d_2}$.

Assume w.l.o.g. $d_1 > d_2$. Then we can almost surely recover Z_2 via

$$Z_2 = \left\{ t \in Z_1 \cup Z_2 : \dim_H((Z_1 \cup Z_2) \cap (s,s')) = 1 - \frac{d_2}{2} \forall s, s' \in \mathbb{Q}, s < t < s' \right\}.$$

Further, (13) and $d_1 > \frac{d_1+d_2}{2} - 1$ show that $Z_1 \setminus Z_2$ must be dense near every point of Z_1. This gives $Z_1 = \overline{(Z_1 \cup Z_2) \setminus Z_2}$.

We conclude that the distribution of $Z_1 \cup Z_2$ is measure isomorphic to the distribution of (Z_1, Z_2) or $\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2} \cong \mathscr{E}^{\mathscr{M}_{d_1} * \mathscr{M}_{d_2}}$.

Moreover, every $\mathscr{M}_{d_1} * \mathscr{M}_{d_2}$-local stationary opening φ induces an $\mathscr{M}_{d_1}$-local stationary opening φ_1 and an $\mathscr{M}_{d_2}$-local stationary opening φ_2 if one of the two sets is empty. That means for $\mathscr{M}_{d_1}$−a.a. Z_1 and $\mathscr{M}_{d_2}$-a.a. Z_2

$$\varphi(Z_1 \cup \emptyset) = \varphi_1(Z_1) \qquad \text{and} \qquad \varphi(\emptyset \cup Z_2) = \varphi_2(Z_2).$$

By Proposition 8.2, each of the maps φ_1 and φ_2 is either almost surely the identity or almost surely constant to the empty set.

If $\varphi_1(Z) = \emptyset$ for all Z, locality implies $\varphi(Z_1 \cup Z_2) \cap (s,t) = \emptyset$ for all s,t such that $Z_2 \cap (s,t) = \emptyset$. If additionally $\varphi_2(Z) = \emptyset$ for almost all Z, we see $\varphi(Z_1 \cup Z_2) \subseteq$

$Z_1 \cap Z_2$. Now observe that $Z_1 \cap Z_2$ is the set of zeros of $(X_t^{(d_1)}, X_t^{(d_2)})_{t\geq 0}$. Applying Proposition 8.3 again yields either $\varphi(Z_1 \cup Z_2) = Z_1 \cap Z_2$ or $\varphi(Z_1 \cup Z_2) = \emptyset$ almost surely. In the former case, we obtain the subsystem $\mathscr{E}^{d_1} \otimes \mathscr{E}^{d_2}$. In the latter case we find the subsystem $\mathscr{E}^{d_1} \otimes_{\mathscr{U}} \mathscr{E}^{d_2}$.

If for almost all Z $\varphi_1(Z) = \emptyset$ and $\varphi_2(Z) = Z$ then $\varphi(Z_1 \cup Z_2) \cap (s,t) = Z_2 \cap (s,t)$ if $Z_1 \cap (s,t) = \emptyset$ such that $\varphi(Z_1 \cup Z_2) = Z_2$. The subsystem must be $\mathscr{E}^{d_1} \otimes v$.

Similar arguments work for $\varphi_2(Z) = \emptyset$ giving the subsystem $u \otimes \mathscr{E}^{d_2}$.

It remains the case $\varphi_1(Z) = \varphi_2(Z) = Z$. Then monotonicity implies $\varphi(Z_1 \cup Z_2) \supseteq \varphi(Z_1 \cup \emptyset) = Z_1$ and, equally, $\varphi(Z_1 \cup Z_2) \supseteq Z_2$ such that $\varphi(Z) = Z$. The subsystem is $u \otimes v$.

Therefore, only the 5 listed subsystems are possible. Theorem 9.1 gives the assertion. ∎

Remark 9 *This result is very similar to [21, Theorem 3.5]. But there only the "diagonal" case $d_1 = d_2$ is considered. The only formal difference we see is the use of all positive contractive cocycles as invariant, whereas we deal with projection valued cocycles (corresponding to Arveson subsystems). In our examples, the space of nontrivial positive contractive cocycles of $\mathscr{E}^d$ is one-dimensional, [21] gives at least an estimate of dimension 2. This indicates that the two examples are nonisomorphic. But, different from us, [21] does not compute all subsystems. Of course it would be quite interesting to translate the QP-flows used by [21] and others into the random-set picture by computing their Arveson system.*

To complete the picture a bit more, we present a slightly surprising result in the diagonal case.

The "diagonal case" $d_1 = d_2 = d$ is more involved since we cannot transform the situation into a question involving one random set in $[0,1]$. We need direct integrals dealing with the multiplicity issue of representations of abelian von Neumann algebras, here $L^\infty(\mathscr{M})$ for the measure type $\mathscr{M}$ coming from embedding $\mathscr{G} = u \otimes u \subset \mathscr{E}$, see [18, section 6]. This theory gives us

$$\mathscr{E}_t = \int^{\oplus} \mu(\mathrm{d}Z) H_Z^t$$

for a measurable family of Hilbert spaces $(H_Z^t)_{Z \in \mathscr{C}_{[0,1]}}$ and some $\mu \in \mathscr{M}_{0,t}$. But, also the change of measures and the product of the Arveson system should play a rôle.

For general embeddings $\mathscr{G} \subseteq \mathscr{E}$, we look at a measurable family of Hilbert spaces $H = (H_Z^t)_{t\geq 0, Z \in \mathscr{C}_{[0,t]}}$ with

1. $H_\emptyset^t = \mathscr{G}_t$ for all $t \geq 0$.

2. There are unitaries $(V^{s,t}_{Z,Z'})_{s,t\geq 0, Z\in\mathscr{C}_{[0,s]}, Z'\in\mathscr{C}_{[0,t]}}$, $V^{s,t}_{Z,Z'}\colon H^s_Z\otimes H^t_{Z'}\mapsto H^{s+t}_{Z\cup(Z'+s)}$ which fulfil the associativity condition

$$V^{r,s+t}_{Z,Z'\cup(Z''+s)}\circ(\mathbf{1}_{H^r_Z}\otimes V^{s,t}_{Z',Z''}) = V^{r+s,t}_{Z\cup(Z'+r),Z''}\circ(V^{r,s}_{Z\cup(Z'+r)}\otimes\mathbf{1}_{H^t_{Z''}}) \tag{14}$$

for all $r,s,t\geq 0$ and for $\mathscr{M}_{0,r}$-a.a. Z, $\mathscr{M}_{0,s}$-a.a. Z' and $\mathscr{M}_{0,t}$-a.a. Z''.

Define a family $\mathscr{F}=(\mathscr{F}_t)_{t\geq 0}$ of Hilbert spaces,

$$\mathscr{F}_t=\left\{\psi=(\psi_\mu)_{\mu\in\mathscr{M}_{0,t}}\colon \psi_\mu\in\int^{\oplus}\mu(\mathrm{d}Z)H^t_Z, \psi_{\mu'}=U_{\mu,\mu'}\psi_\mu\,\forall\mu,\mu'\in\mathscr{M}_{0,t}\right\}.$$

Again, the unitaries $U_{\mu,\mu'}$ are given through (9). $\mathscr{F}$ is equipped with product unitaries $(W_{s,t})_{s,t\geq 0}$, $W_{s,t}\colon \mathscr{F}_s\otimes\mathscr{F}_t\mapsto\mathscr{F}_{s+t}$, given through

$$(W_{s,t}\psi\otimes\tilde{\psi}')_{\mu_{0,s}\otimes(\mu'_{0,t}+s)}(Z\cup(Z'+s)) = V^{s,t}_{Z,Z'}\psi_{\mu_{0,s}}(Z)\otimes\psi'_{\mu'_{0,t}}(Z').$$

Then $\mathscr{F}$ is an Arveson system, see [18, Lemma 6.6], denote it by $\mathscr{E}^{\mathscr{M},H}$.

We need the following result

Theorem 9.3 ([18, Theorem 6.7]) *Let $\mathscr{E}$ be an Arveson system, $\mathscr{G}\subseteq\mathscr{E}$ a subsystem and $\mathscr{M}=\mathscr{M}^{\mathscr{G}}$ the corresponding measure type.*

Then there exists a measurable family of Hilbert spaces $H=(H^t_Z)_{t\geq 0, Z\in\mathscr{C}_{[0,t]}}$ such that $\mathscr{E}\cong\mathscr{E}^{\mathscr{M},H}$ under an isomorphism respecting the natural actions of $J_{\mathscr{G}}(L^\infty(\mathscr{M}^{\mathscr{G}}))$ and $L^\infty(\mathscr{M}^{\mathscr{G}})$.

For the next result, let $\mathbb{P}^1_{\mathbb{C}}$ denote the one-dimensional complex projective space, i.e. the space of all one-dimensional subspaces of $\mathbb{C}^2$.

Theorem 9.4 *Suppose $d\geq 1$.*

Then $\mathscr{M}_d*\mathscr{M}_d=\mathscr{M}_d$ and thus $\mathscr{E}^d\otimes\mathscr{E}^d\ncong\mathscr{E}^{\mathscr{M}_d*\mathscr{M}_d}$.

Moreover, $\mathscr{E}^d\otimes\mathscr{E}^d$ has infinitely many proper subsystems: 0, $u\otimes u$, and a continuum $(\mathscr{E}^z)_{z\in\mathbb{P}^1_{\mathbb{C}}}$ of subsystems isomorphic to $\mathscr{E}^d$. Thus, $\mathscr{S}(\mathscr{E}^d\otimes\mathscr{E}^d)$ has depth 4.

Proof. From Theorem 9.1 we know under $\mathscr{M}_d\otimes\mathscr{M}_d$ that $Z_1\cap Z_2=\emptyset$ a.s. Therefore, [18, Proposition 4.20] shows $\mathscr{M}_d*\mathscr{M}_d=\mathscr{M}_d$. This gives us another view on the random set $(Z_1,Z_2)\sim\mathscr{M}_d\otimes\mathscr{M}_d$ underlying the Arveson system $\mathscr{E}^d\otimes\mathscr{E}^d$: we could condition on $Z=Z_1\cup Z_2$. If $\mathscr{L}(Z_1),\mathscr{L}(Z_2)=\mu$, we obtain the conditional distribution of the pair (Z_1,Z_2) given $Z_1\cup Z_2=Z$ as a stochastic kernel $q_\mu(\cdot|Z)$.

Let us consider the direct integral representation of $\mathscr{E}$. We derive it by disintegration with respect to the measure $\mu*\mu\in\mathscr{M}_d$, i.e.

$$(\mathscr{E}^d\otimes\mathscr{E}^d)_t=\int_{\oplus}\mu*\mu(\mathrm{d}Z)H^t_Z$$

with

$$H_Z^t = L^2(q_\mu(\cdot|Z)).$$

Observe that for $\mu, \mu' \in \mathscr{M}_d$ the conditional distributions $q_\mu(\cdot|Z)$ and $q_{\mu'}(\cdot|Z)$ are equivalent for almost all Z.

Since $Z_1 \cap Z_2 = \emptyset$ a.s., there is a partition $\mathfrak{t} = (t_1, \ldots, t_n) \in \Pi^1$ such that for all i either $Z_1 \cap [t_1 + \cdots + t_{i-1}, t_1 + \cdots + t_i] = \emptyset$ or $Z_2 \cap [t_1 + \cdots + t_{i-1}, t_1 + \cdots + t_i] = \emptyset$. We describe this situation by $(Z_1, Z_2) \to \mathfrak{t}$. We could even choose the $t_i \in \mathbb{Q}$. Thus there are only countably many choices of the partitions and pairs (Z_1, Z_2) compatible with $Z_1 \cup Z_2 = Z$. We conclude that $q_\mu(\cdot|Z)$ is a discrete measure. Further, $q_\mu(\{(Z_1, Z_2)\}|Z) = q_\mu(\{(Z_2, Z_1)\}|Z)$ since $\mu \otimes \mu$ is symmetric.

Now any Arveson subsystem $\mathscr{G}$, $u \otimes u \subset \mathscr{G} \subset \mathscr{E}$ is determined by Hilbert spaces $H'_{t,Z} \subseteq H_{t,Z}$ sharing the tensor products from the family H. We introduce now the spaces

$$G_{t,Z} = \left\{ \begin{pmatrix} \psi(Z, \emptyset) \\ \psi(\emptyset, Z) \end{pmatrix} : \psi \in H'_{t,Z}) \right\} \subseteq \mathbb{C}^2,$$

$t \in [0,1]$, $Z \in \mathscr{C}_{[0,t]}$. By symmetry of $\mu \otimes \mu$, these spaces are independent from the choice of the measure $\mu \in \mathscr{M}_d$:

$$\begin{aligned} U_{\mu\otimes\mu,\mu'\otimes\mu'}\psi(Z,\emptyset) &= \sqrt{\frac{q_{\mu'}(\{(Z,\emptyset)\}|Z)}{q_\mu(\{(Z,\emptyset)\}|Z)}}\psi(Z,\emptyset) \\ U_{\mu\otimes\mu,\mu'\otimes\mu'}\psi(\emptyset,Z) &= \sqrt{\frac{q_{\mu'}(\{(\emptyset,Z)\}|Z)}{q_\mu(\{(\emptyset,Z)\}|Z)}}\psi(\emptyset,Z) = \sqrt{\frac{q_{\mu'}(\{(Z,\emptyset)\}|Z)}{q_\mu(\{(Z,\emptyset)\}|Z)}}\psi(\emptyset,Z). \end{aligned}$$

It is easy to see that the family H' is uniquely determined by G. For, consider Z distributed according to $\mathscr{M}_d$ and a partition $\mathfrak{t} \in \Pi^1$ like mentioned above. Then

$$H'_{1,Z} = H'_{t_1,Z_{0,t_1}} \otimes H'_{t_2,Z_{t_1,t_1+t_2}-t_1} \otimes \cdots \otimes H'_{t_n,Z_{t_1+\cdots+t_{n-1},t_1+\cdots+t_n}-(t_1+\cdots+t_{n-1})}$$

So

$$\left\{(\psi(Z_1,Z_2))_{(Z_1,Z_2)\to\mathfrak{t}} : \psi \in H'_{1,Z}\right\} \subseteq \mathbb{C}^{2^n},$$

is fixed. Varying $\mathfrak{t}$, we find that $H'_{1,Z}$ and consequently all $H'_{t,Z}$ are fixed by $G_{t,Z}$.

How does $G_{t,Z}$ depend on t and Z? Of course, $G_{1,Z} = G_{1,Z+s}$ for all s, since $\mathscr{G}$ is a subsystem. Consider first

$$Q = \left\{ Z \in \mathscr{C}_{[0,1]} : \begin{pmatrix} \mathbb{C} \\ 1 \end{pmatrix} \cap G_{1,Z} \neq \emptyset \right\}.$$

It is easy to see that $Z \in Q$ if and only if $(Z_{0,t} \in Q) \wedge (Z_{t,1} \in Q)$. Also, $Z \in Q$ if and only if $Z + t \in Q$ $\mathscr{M}_d$-a.s. Thus 1_Q is the projection onto $\mathscr{F}_1$ for a subsystem of $\mathscr{E}^d$.

By Theorem 8.1 those are trivial and $\emptyset \in Q$, thus either $Q = \{\emptyset\}$ or $Q = \mathscr{C}_{[0,1]}$ almost surely. A similar argument applies to $Q' = \left\{ Z \in \mathscr{C}_{[0,1]} : \begin{pmatrix} 1 \\ \mathbb{C} \end{pmatrix} \cap G_{1,Z} \neq \emptyset \right\}$.

If both $Q = Q' = \{\emptyset\}$, $G_{t,Z} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ unless $Z = \emptyset$. This means $\mathscr{G} = u \otimes u$.

If $Q = \{\emptyset\}$, $Q' = \mathscr{C}_{[0,1]}$, $G_{t,Z} = \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ unless $Z = \emptyset$. This means

$$H'_{t,Z} = \{ \psi : \psi(Z_1, Z_2) = 0 \text{ unless } Z_1 = \emptyset \}$$

The case $Q' = \{\emptyset\}$ and $Q = \mathscr{C}_{[0,1]}$ is discussed similarly.

Now suppose $Q = Q' = \mathscr{C}_{[0,1]}$, Let

$$Q'' = \left\{ Z \in \mathscr{C}_{[0,1]} : \dim G_{1,Z} = 1 \right\}.$$

Again we see that $Z \in Q''$ if and only if $(Z_{0,t} \in Q'') \wedge (Z_{t,1} \in Q'')$. Furthermore, $Z \in Q''$ if and only if $Z + t \in Q''$ $\mathscr{M}_d$-a.s. So there are two possibilities: If $Q'' = \{\emptyset\}$, $G_{t,Z} = \mathbb{C}^2$ unless $Z = \emptyset$. This means $H'_{t,Z} = H_{t,Z}$. Otherwise, $Q'' = \mathscr{C}_{[0,1]}$ implies there is some $\lambda(t,Z) \in \mathbb{C} \setminus \{0\}$ such that $G_{t,Z} = \mathbb{C} \begin{pmatrix} 1 \\ \lambda(t,Z) \end{pmatrix}$. Of course, λ is stationary and fulfils almost surely

$$\lambda(1,Z) = \lambda(t, Z_{0,t}) \lambda(1-t, Z_{t,1} - t)$$

Since λ has to be measurable and $\lambda(t,\emptyset) = 1$, we find similar to Theorem 8.2 some $w \in \mathbb{C}$ such that

$$\lambda(t,Z) = \mathrm{e}^{w L_t(Z)}$$

This completes the proof. ∎

Remark 10 *Clearly, the gauge group of $\mathscr{E}^d \otimes \mathscr{E}^d$ is nontrivial. Nevertheless, this is already for $\mathscr{E}^d$ the case, see Theorem 8.2. Nevertheless, the gauge group of $\mathscr{E}^d \otimes \mathscr{E}^d$ is even not the direct square of the gauge groups. This resembles the type I case. Loosely speaking, we would classify*

$$\begin{array}{ll} \mathscr{E}^d & \text{to be of type } \mathrm{II}_{\mathscr{M}_d,0} \\ \mathscr{E}^d \otimes \mathscr{E}^d & \text{to be of type } \mathrm{II}_{\mathscr{M}_d,1} \\ \mathscr{E}^d \otimes \mathscr{E}^d \otimes \mathscr{E}^d & \text{to be of type } \mathrm{II}_{\mathscr{M}_d,2} \\ & \vdots \end{array}$$

We derive here some kind of conditional index. It is given by $\dim G_{t,Z} - 1$*, which is essentially independent of t and Z as shown in the proof above. Even more, the sections $u^w_{t,Z}$,*

$$u^w_{t,Z,\mu}(Z_1, Z_2) = c_{t,Z,\mu} \mathrm{e}^{w L_t(Z_1)}$$

play the rôle of "conditional units". For instance, for carefully chosen constants $c_{t,Z,\mu}$ *they fulfil*

$$u^w_{s,Z} \otimes u^w_{t,Z'} = u^w_{s+t,Z\cup(Z'+s)}.$$

Remark 11 *This similarity with the Arveson system of type* I_1 *or Arveson systems of type* II_1 *as constructed from [18, section 4.3] gives rise to the following question:*

> *Let* $\mathscr{G}_1, \mathscr{G}_2$ *be two different but isomorphic subsystems of an Arveson system* $\mathscr{E}$.
>
> *Does there exist a* $\mathbb{P}^1_{\mathbb{C}}$*-parametrised family of mutually different subsystems of* $\mathscr{E}$*, all of which are isomorphic to* $\mathscr{G}_1, \mathscr{G}_2$*?*

Remark 12 *If* $d < 1$*, we expect a more complicated structure and* $\mathscr{S}(\mathscr{E}^d \otimes \mathscr{E}^d)$ *to have again depth 5. Surely, there is a chain of length 5, but now we are not so sure about the subsystems "between"* $\mathscr{E}^d \otimes_{\mathscr{U}} \mathscr{E}^d$ *and* $\mathscr{E}^d \otimes \mathscr{E}^d$*. At least there seem to be parallels to type* II_1 *Arveson systems.*

Observe that the analysis of the spatial *product* $\mathscr{E}^d \otimes_{\mathscr{U}} \mathscr{E}^d$ *remains unchanged from the case* $d \geq 1$.

References

[1] L. Accardi, On square roots of measures, In: *C*-Algebr. Appl. Stat. Mech. Quantum Field Theory, Proc. Int. Sch. Phys. "Enrico Fermi", Varenna, 1973*, 1976, pp. 167–189

[2] A. Alevras, R. T. Powers, and G. L. Price, Cocycles for one-parameter flows on $\mathscr{B}(\mathscr{H})$, *J.Func.Anal.* **230**: 1–64, 2006

[3] W. Arveson, Continuous analogues of Fock spaces, *Mem.Americ.Math.Soc.* **80**, 1989

[4] W. Arveson, *Noncommutative dynamics and E-semigroups*, Springer, 2003

[5] J. Bertoin, Subordinators: Examples and Applications , In: *Ecole d'été de Prob-abilités de St-Flour XXVII*, Lecture Notes in Math. 1717, Springer, Berlin, 1999

[6] B.V.R. Bhat, Cocycles of CCR flows, *Mem.Americ.Math.Soc.* **709**, American Mathematical Society, 2000

[7] B.V.R. Bhat, An index theory for quantum dynamical semigroups, *Trans. Amer. Math. Soc.* **348** : 561–583, 1996

[8] B.V.R. Bhat, V. Liebscher, and M. Skeide, A Problem of Powers and the Product of Spatial Product Systems, In: *Proceedings of the 27th Conference on Infinite Dimensional Analysis*, Quantum Probability and White Noise Analysis XXIII, pages 93-106, World Scientific, 2008

[9] B.V.R. Bhat and M. Mukherjee, Inclusion systems and amalgamated products of product systems, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13**, 1–26, 2010, (arXiv: 0907.0095v1)

[10] B.V.R. Bhat, V. Liebscher, M. Mukherjee, and M. Skeide, The Spatial Product of Arveson Systems is Intrinsic, *J. Func. Analysis*, **260**(2):566–573, 2011, (arXiv: 1006.2746v1)

[11] B.V.R. Bhat, V. Liebscher, and M. Skeide, Subsystems of Fock need not be Fock: Spatial CP-semigroups, *Proc. Amer. Math. Soc.* **138**: 2443–2456, 2010, (arXiv: 0804.2169v2)

[12] B.V.R. Bhat, M. Lindsay, and M. Mukherjee, Additive units of product system, preprint, 2015, (arXiv: 1501.07675v1)

[13] B.E. Fristedt and W.E. Pruitt, Lower functions for increasing random walks and subordinators, *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **18**: 167–182, 1971

[14] H.J. Heijmans, *Morphological image operators*, Acad. Press Boston etc., 1994

[15] C. Jankowski, D. Markiewicz, and R.T. Powers, Aligned CP-semigroups, *Int.Math.Res.Notices* **15**: 6639-6647, 2015 (arXiv: 1107.2053v1)

[16] J.-P. Kahane, Sur la dimension des intersections, In: *Aspects of mathematics and its applications. (Collection of papers written in honour of Leopoldo Nachbin).* North-Holland, Amsterdam, 1986, pp. 419–430

[17] O. Kallenberg, *Foundations of Modern Probability*, 2nd edition Springer, Berlin etc., 2001

[18] V. Liebscher, Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces, *Mem.Amer.Math.Soc.* **930**, 2009, (arXiv: math.PR/0306365)

[19] D. Markiewicz, R.T. Powers, Local unitary cocycles of E_0-semigroups, *J.Funct.Anal.* **256** (5): 1511–1543, 2009

[20] M. Mukherjee, On cluster systems of tensor product systems of Hilbert spaces, preprint, 2015 (arXiv: 1502.03247v1)

[21] R.T. Powers, Addition of spatial E_0–semigroups, In: *Operator algebras, quantization, and noncommutative geometry*, Contemporary Mathematics, no. 365, American Mathematical Society, 2004, pp. 281–298

[22] M. Skeide, Commutants of von Neumann modules, representations of $\mathscr{B}^a(E)$ and other topics related to product systems of Hilbert modules, In: *Advances in quantum dynamics (G.L. Price, B .M. Baker, P.E.T. Jorgensen, and P.S. Muhly, eds.)*, Contemporary Mathematics, no. 335, American Mathematical Society, 2003, pp. 253–262, (arXiv: math.OA/0308231)

[23] M. Skeide, The index of (white) noises and their product systems, *Infin.Dimens.Anal.Quantum Probab.Relat.Top.* **9**, 617–655, 2006, (arXiv: math.OA/0601228)

[24] M. Skeide, The Powers sum of spatial CPD-semigroups and CP-semigroups, *Banach Center Publications* **89**: 247–263, 2010, (arXiv: 0812.0077)

[25] D. Revuz and M. Yor, *Continuous Martingals and Brownian Motion*, Springer, Berlin Heidelberg 1991

[26] T. Shiga and S. Watanabe, Bessel diffusions as a one-parameter family of diffusion processes, *Z. Wahrscheinlichkeitsth. Verw. Gebiete* **27**: 34-46, 1973

[27] B. Tsirelson, Unitary Brownian motions are linearizable, preprint 1998, (arXiv:math.PR/9806112v1)

[28] B. Tsirelson, From random sets to continuous tensor products: answers to three questions of W. Arveson, preprint 2000 (arXiv: math/0001070)

[29] B. Tsirelson, Non-Isomorphic product systems, In: *Contemporary mathematics, AMS*, 2003, Contemporary mathematics, (arXiv: math.FA/0210457)

[30] B. Tsirelson, Automorphisms of Type II Arveson systems (probabilistic approach), *New York Journal of Mathematics* **14**: 539–576, 2008

[31] J.B. Walsh, The perfection of multiplicative functionals, In: *Sem. Probab. 6, Univ. Strasbourg 1970-71*, Lect. Notes Math. 258, 233–242, 1972