

Multivariate Complex B-Splines

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ABSTRACT

We extend the notion of complex B-splines to a multivariate setting by employing the relationship between ordinary B-splines and multivariate B-splines by means of ridge functions. In order to obtain properties of complex B-splines in $\mathbb{R}^s$, $1 < s \in \mathbb{N}$, the Dirichlet average has to be generalized to include infinite dimensional simplices. Based on this generalization several identities of multivariate complex B-splines are exhibited.

Keywords: Complex splines, multivariate splines, Dirichlet average, Weyl fractional derivative and integral.

1. INTRODUCTION

Recently, a complex variant B_z of B-splines was defined in Forster et al.¹ For $\operatorname{Re} z \geq 1$, define

$$B_z(x) := \frac{1}{\Gamma(z)} \sum_{k \geq 0} (-1)^k \binom{z}{k} (x - k)_+^{z-1}, \quad (1)$$

where

$$x_+^z = \begin{cases} x^z = e^{z \ln x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The series (1) converges for all $x \in \mathbb{R}$. It has been shown that, for fixed z with $\operatorname{Re} z > 1$, the functions B_z are elements of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and that their Fourier transform is given by

$$\mathcal{F}(B_z)(\omega) = \int_{\mathbb{R}} B_z(x) e^{-i\omega x} dx = \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^z.$$

It thus follows that

$$\int_{\mathbb{R}} B_z(x) dx = \mathcal{F}(B_z)(0) = 1. \quad (2)$$

For $z = n \in \mathbb{N}$, the complex B-spline B_z reduces to the classical Curry-Schoenberg B-spline B_n , $n \in \mathbb{N}$. (Cf. Curry & Schoenberg.²) For an interpretation of the complex B-spline in the Fourier domain, we set $\Omega(\omega) := \frac{1 - e^{-i\omega}}{i\omega}$. Then

$$\mathcal{F}(B_z)(\omega) = (\Omega(\omega))^z = (\Omega(\omega))^{\operatorname{Re} z} e^{i \operatorname{Im} z \ln |\Omega(\omega)|} e^{-\operatorname{Im} z \arg \Omega(\omega)} = \mathcal{F}(B_{\operatorname{Re} z})(\omega) e^{i \operatorname{Im} z \ln |\Omega(\omega)|} e^{-\operatorname{Im} z \arg \Omega(\omega)},$$

i.e., a complex B-spline is a fractional B-spline of order $\operatorname{Re} z$ with a phase and modulation/scaling factor in Fourier domain. Note that because of $\arg \Omega(\omega) \geq 0$, the frequency components on the negative and positive axis are enhanced by the opposite sign. The fractional as well as the complex B-splines are scaling functions for multiresolution analyses of $L^2(\mathbb{R})$. (See Unser & Blu³ and Forster et al.¹)

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The following relations between complex B-splines and divided differences of order $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ hold. (See also Forster & Massopust.⁴) Let $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ be a sequence of uniform knots. We define the corresponding complex divided difference operator as

$$[z; \mathbb{N}_0]g := \sum_{k \geq 0} (-1)^k \frac{g(k)}{\Gamma(z - k + 1)\Gamma(k + 1)} \quad (3)$$

for all functions $g : \mathbb{R} \rightarrow \mathbb{C}$ with convergent series on the right hand side. Then

$$B_z(x) = z[z; \mathbb{N}_0](\bullet - k)_+^{z-1}. \quad (4)$$

For $z = n \in \mathbb{N}_0$, equations (3) and (up to a factor $(-1)^n$) (4) reduce to the standard forms

$$[t_0, \dots, t_n]g = \sum_{j=0}^n \frac{g(t_j)}{\prod_{l \neq j} (t_j - t_l)}.$$

for the finite sequence of knots $\{t_0, \dots, t_n\} = \{0, \dots, n\}$ and

$$B_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^n (-1)^k \binom{n}{k} (x - k)_+^{n-1} = (-1)^n n[0, 1, \dots, n](\bullet - k)_+^{n-1}. \quad (5)$$

In this article, we will derive a multidimensional extension of the complex B-splines.

2. MULTIVARIATE COMPLEX B-SPLINES

For the n -th order B-spline B_n , $n \in \mathbb{N}$, the following relation is well-known:

$$[0, 1, \dots, n]g = \frac{1}{n!} \int_{\mathbb{R}} B_n(t) g^{(n)}(t) dt. \quad (6)$$

An analogue for complex B-splines reads as follows. (Cf. Forster & Massopust.⁴)

Proposition 1. Let $\operatorname{Re} z > 1$ and $g \in \mathcal{S}(\mathbb{R})$ belong to the Schwartz space. Then the complex B-splines B_z and the complex divided differences (3) satisfy the following relation.

$$[z; \mathbb{N}_0]g = \frac{1}{\Gamma(z)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t) dt,$$

where $g^{(z)} := W^z g$ denotes the Weyl fractional derivative (cf. Kilbas et al.⁵) of order z :

$$W^z g(t) = (-1)^n \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(\nu)} \int_t^\infty (x - t)^{\nu-1} g(x) dx \right],$$

with $n = \lceil \operatorname{Re} z \rceil$, and $\nu = n - z$. Here $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$, $x \mapsto \min\{n \in \mathbb{Z} \mid n \geq x\}$, denotes the *ceiling function*. As a convention, $W^0 := \operatorname{id}_{\mathcal{S}(\mathbb{R})}$.

We can extend complex B-splines to include arbitrary weights b and arbitrary sequences of knots. To this end, let Δ^∞ be the infinite dimensional standard simplex $\Delta^\infty := \left\{ u := (u_j)_j \in (\mathbb{R}_0^+)^{\mathbb{N}_0} \mid \sum_{j=0}^\infty u_j = 1 \right\}$, endowed with the topology of pointwise convergence, i.e., the weak-* topology. Let $\mu_b = \varprojlim \mu_b^n$ denote the projective limit of Dirichlet measures on the n -dimensional standard simplex Δ^n with density $\frac{\Gamma(b_0) \cdots \Gamma(b_n)}{\Gamma(b_0 + \cdots + b_n)} u_0^{b_0-1} u_1^{b_1-1} \cdots u_n^{b_n-1}$.

Definition 2. Given a weight vector $b \in \mathbb{R}_+^\infty$ and an increasing knot sequence $\tau := \{t_k\}_k \in \mathbb{R}^{\mathbb{N}_0}$ with the property that $\lim_{k \rightarrow \infty} \sqrt[k]{t_k} < \infty$. A complex B-spline $B_z(\bullet \mid b; \tau)$ with weight vector b and knot sequence τ is a function satisfying

$$\int_{\mathbb{R}} B_z(t \mid b; \tau) g^{(z)}(t) dt = \int_{\Delta^\infty} g^{(z)}(\tau \cdot u) d\mu_b(u) \quad (7)$$

for all $g \in \mathcal{S}^\omega(\mathbb{R})$. (Here, $\mathcal{S}^\omega(\mathbb{R}) := \mathcal{S}(\mathbb{R}) \cap C^\omega(\mathbb{R})$, with $C^\omega(\mathbb{R})$ denoting the real-analytic functions on $\mathbb{R}$, and $\tau \cdot u = \sum_{k \in \mathbb{N}} t_k u_k$ for $u = \{u_k\}_{k \in \mathbb{N}} \in \Delta^\infty$.)

Remark 3. For finite $\tau = \tau(n)$ and $b = b(n)$, (7) defines also the so-called *Dirichlet splines* if g is chosen in $C^n(\mathbb{R})$. For, Dirichlet splines $D(\cdot | b; \tau)$ are defined as those functions for which

$$\int_{\mathbb{R}} g^{(n)}(t) D(t | b; \tau) dt = \int_{\Delta^n} g^{(n)}(\tau \cdot u) d\mu_b(u) = G^{(n)}(b; \tau), \quad \tau \in \mathbb{R}^{n+1},$$

holds true for all $g \in C^n(\mathbb{R})$ and thus for $g \in \mathcal{S}^\omega(\mathbb{R})$. (Here G is the Dirichlet average of g .)

As an analog to (6) we define divided differences of g of order z for the sequence of knots τ as

$$[z; \tau]g := \int_{\mathbb{R}} B_z(t | b; \tau) g^{(z)}(t) dt, \quad \text{for all } g \in \mathcal{S}(\mathbb{R}). \quad (8)$$

We extend the notion of complex B-splines to a multivariate setting in $\mathbb{R}^s$, $s \geq 1$, via the notion of ridge functions. (See, for instance, Pinkus.⁶) This approach has already led to an extension of the Curry–Schoenberg-splines to a multivariate setting. (Cf. Micchelli⁷ and Neuman & Van Fleet.⁸) Let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be a direction, and let $g : \mathbb{R} \rightarrow \mathbb{C}$ be some function. The corresponding ridge function is defined as $g_\lambda : \mathbb{R}^s \rightarrow \mathbb{C}$, $g_\lambda(x) = g(\langle \lambda, x \rangle)$ for all $x \in \mathbb{R}^s$. (Here, $\langle \bullet, \bullet \rangle$ denotes the canonical inner product in Euclidean space.)

Definition 4. Let $\tau = \{t^n\}_{n \in \mathbb{N}_0} \in (\mathbb{R}^s)^{\mathbb{N}_0}$ be a sequence of knots in $\mathbb{R}^s$ satisfying $\limsup_{n \rightarrow \infty} \sqrt[n]{\|t^n\|} < \infty$. We define the multivariate complex B-spline $B_z(\bullet | b, \tau) : \mathbb{R}^s \rightarrow \mathbb{C}$ for the weight vector b via

$$\int_{\mathbb{R}^s} g^{(z)}(\langle \lambda, x \rangle) B_z(x | b, \tau) dx = \int_{\mathbb{R}} g^{(z)}(t) B_z(t | b, \lambda \tau) dt \quad (9)$$

for all $g \in \mathcal{S}(\mathbb{R})$, and where we defined $\lambda \tau := \{\langle \lambda, t^n \rangle\}_{n \in \mathbb{N}}$.

Convention. We will index the elements of a collection of vectors in $\mathbb{R}^s$ or $\mathbb{C}^s$ by superscripts and their components by subscripts, i.e., if $T := \{t^1, \dots, t^n\}$ is a collection of vectors in $\mathbb{C}^s$ then t_j^k will denote the j -th component of the k -th vector in T .

As the knot set τ depends on z , we write $\tau = \tau(z)$ and note that $\tau(z) = \mathbb{N}_0$ for $z \in \mathbb{C} \setminus \mathbb{N}_0$ and $\tau(z) = \mathbb{N}_0^n$, for $z \in \mathbb{N}$, where $\mathbb{N}_0^n := \{0, 1, \dots, n\}$ denotes the initial segment of $\mathbb{N}_0$ of length $n + 1$. Setting $z := n \in \mathbb{N}$ in (9), the infinite sequences b and τ collapse to $b(n) := (b_0, b_1, \dots, b_n, 0, 0, \dots)$ and $\tau(n) := (t^1, \dots, t^n, 0, 0, \dots)$ and (9) becomes a well-known relation between univariate and multivariate B-splines. (Cf. Karlin et al.⁹ and Micchelli.⁷)

For the special case $b = e = (1, 1, 1, \dots)$, the multivariate divided differences of order z are defined on ridge functions via

$$\begin{aligned} [z; \tau]g_\lambda &= [z; \tau]g(\langle \lambda, \bullet \rangle) = \frac{1}{\Gamma(z)} \int_{\mathbb{R}^s} g^{(z)}(\langle \lambda, x \rangle) B_z(x | e, \tau) dx \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}} g^{(z)}(t) B_z(t | e, \lambda \tau) dt = [z; \lambda \tau]g, \quad \forall \lambda \in \mathbb{R}^s; \forall g \in \mathcal{S}(\mathbb{R}^\infty). \end{aligned}$$

Ridge functions form a dense subset of $C(\mathbb{R}^k)$, $k \in \mathbb{N}$. (See also Króo.¹⁰) For $n \in \mathbb{N}$ and a finite sequence of knots $\tau = \{t^0, t^1, \dots, t^n\}$, one obtains

$$\begin{aligned} [t^0, \dots, t^n]g_\lambda &= [n; \tau]g(\langle \lambda, \bullet \rangle) = \frac{1}{\Gamma(z)} \int_{\mathbb{R}^s} g^{(n)}(\langle \lambda, x \rangle) B(x | e, \tau) dx \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}} g^{(n)}(t) B_n(t | e, \lambda \tau) dt = [n; \lambda \tau]g = \sum_{j=0}^n \frac{g(\langle \lambda, t^j \rangle)}{\prod_{l \neq j} \langle \lambda, t^j - t^l \rangle}. \end{aligned}$$

In order to obtain some properties of multivariate complex B-splines, we need to introduce *Dirichlet averages*.

3. DIRICHLET AVERAGES

Dirichlet averages are discussed in the book by Carlson¹¹ and related to univariate and multivariate B-splines in Carlson¹², and they have produced deep and interesting connections to special functions. In this section, we extend the notion of Dirichlet average to the infinite-dimensional setting and show that under mild conditions on the weights the results important for our interests do also hold on Δ^∞ . In particular, we show that using a geometric interpretation, the Weyl fractional derivative and integral can be applied to Dirichlet averages.

To this end, let Ω be a convex open subset of $\mathbb{C}^s$, $s \in \mathbb{N}$, let $\zeta \in \bigtimes_{i=0}^n \Omega$, and let $b \in \mathbb{R}_+^{n+1}$. Then the Dirichlet average of a measurable function $f : \Omega \rightarrow \mathbb{C}$ is defined as the integral

$$F(b; \zeta) := \int_{\Delta^n} f(\zeta \cdot u) d\mu_b^n(u), \quad (10)$$

where $u \cdot \zeta := \sum_{i=0}^n u_i \zeta^i \in \mathbb{C}^s$. We note that it is customary to denote the Dirichlet average of a function f by the corresponding upper-case letter, F . It can be shown that the Dirichlet average of a derivative equals the derivative of the Dirichlet average. (For more details regarding the properties of Dirichlet averages and their connection to the theory of special functions, we refer the interested reader to the work by Carlson.¹¹)

Let $\mathbb{C}_> := \{\zeta \in \mathbb{C} \mid \operatorname{Re} \zeta > 0\}$. The following result is known. (Cf. Carlson.¹¹)

Proposition 5. Suppose that $f : \Omega \rightarrow \mathbb{C}$ is holomorph. Then the Dirichlet average $F(\cdot, \zeta)$ is a holomorphic function on $\mathbb{C}_>^{n+1}$, for fixed $\zeta \in \Omega^{n+1}$.

The extension of (10) to Δ^∞ consists of taking Ω to be an open convex set in $\mathbb{C}^s$, $b \in \mathbb{R}_+^{\mathbb{N}_0}$, and choosing a measurable function $f \in \mathcal{S}(\Omega) := \mathcal{S}(\Omega, \mathbb{C})$. For $\zeta \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ and $u \in \Delta^\infty$, define $u \cdot \zeta$ to be the bilinear mapping $(u, \zeta) \mapsto \sum_{i=1}^\infty u_i \zeta^i$. The infinite sum exists whenever $\limsup_{n \rightarrow \infty} \sqrt[n]{\|\zeta^n\|} < \infty$, where $\|\cdot\|$ denotes the canonical Euclidean norm on $\mathbb{C}^s$. (See also Forster & Massopust.⁴)

Definition 6. The Dirichlet average $F : \mathbb{R}_+^{\mathbb{N}_0} \times \Omega^{\mathbb{N}_0} \rightarrow \mathbb{C}$ on Δ^∞ is defined by

$$F(b; \zeta) := \int_{\Delta^\infty} f(u \cdot \zeta) d\mu_b(u),$$

where $\mu_b = \varprojlim \mu_b^n$ is the projective limit of Dirichlet measures on the n -dimensional standard simplex Δ^n .

Under the assumption that $f : \Omega \rightarrow \mathbb{C}$ is a holomorphic function and b satisfies the above condition, the Dirichlet average on Δ^∞ exists and is holomorph on $\mathbb{C}_>^\infty$ for fixed $\zeta \in \Omega^{\mathbb{N}_0}$. Using the fact that Δ^∞ is the projective limit of its finite-dimensional projections Δ^n , $n \in \mathbb{N}$, the following known properties of F extend naturally to the infinite-dimensional setting:

- Let $\sigma : \mathbb{N}_0^\infty \rightarrow \mathbb{N}_0^\infty$ be a permutation. Then

$$F(b_{\sigma(0)}, b_{\sigma(1)}, \dots; \zeta^{\sigma(0)}, \zeta^{\sigma(1)}, \dots) = F(b_0, b_1, \dots; \zeta^0, \zeta^1, \dots);$$

- $F(b_0, b_1, b_2, \dots; \zeta^1, \zeta^1, \zeta^2, \dots) = F(b_0 + b_1, b_2, \dots; \zeta^1, \zeta^2, \dots);$
- $F(0, b_1, b_2, \dots; \zeta^0, \zeta^1, \zeta^2, \dots) = F(b_1, b_2, \dots; \zeta^1, \zeta^2, \dots);$

- If $\zeta = (z, z, z, \dots) \in \Omega^{\mathbb{N}_0}$, then $u \cdot \zeta = z \sum_{i=0}^\infty u_i = z \in \mathbb{C}^s$, and thus $F(b; \zeta) = f(z)$.

Now suppose that the weight vector $b \in \ell^1(\mathbb{N}_0)$. Let $c := \sum_{i=0}^{\infty} b_i$ and $w_i := \frac{b_i}{c}$. Since for $j = 1, \dots, n$, the equation

$$\int_{\Delta^n} u_j d\mu_b^n(u) = \frac{b_j}{\sum_{i=0}^{\infty} b_i}$$

holds for all finite-dimensional projections Δ^n of Δ^∞ (Carlson¹¹), we have

$$\int_{\Delta^\infty} u_j d\mu_b(u) = \frac{b_j}{c} = w_j, \quad \forall j \in \mathbb{N}_0.$$

In a similar fashion, using the fact that (Section 4.4 in Carlson¹¹),

$$u_1^{m_1} \cdots u_k^{m_k} d\mu_b^k(u) = \frac{(\Gamma(b_1 + \cdots + b_k))(\Gamma(b_1 + m_1) \cdots \Gamma(b_k + m_k))}{(\Gamma(b_1) \cdots \Gamma(b_k))(\Gamma(b_1 + m_1 + \cdots + b_k + m_k))} d\mu_{b+m}^k(u),$$

for $m \in \mathbb{N}^k$, one obtains the identity

$$u_j d\mu_b(u) = w_j d\mu_{b+e_j}(u), \quad j \in \mathbb{N}_0, \quad (11)$$

generalizing the corresponding finite-dimensional identity. (Cf. Carlson.¹¹) (Here $e_j := \{\delta_{i,j} \mid i \in \mathbb{N}_0\}$.) From (11) one obtains the identity

$$\int_{\Delta^\infty} f(u \cdot \zeta) d\mu_b(u) = \sum_{j=0}^{\infty} w_j \int_{\Delta^\infty} f(u \cdot \zeta) d\mu_{b+e_j}(u),$$

or, equivalently,

$$F(b; \zeta) = \sum_{j=0}^{\infty} w_j F(b + e_j; \zeta).$$

In particular, for $x \in \mathbb{R}^s$ and $g(x) := xf(x)$, this last equation gives

$$G(b; \zeta) = \sum_{j=0}^{\infty} w_j \zeta^j F(b + e_j; \zeta),$$

where $\zeta^j \in \mathbb{C}^s$ is the j th component of ζ .

The results regarding the relations between Dirichlet averages found in Carlson,¹² Section 5, or Neuman & Van Fleet,⁸ Section 3, transfer to the infinite-dimensional setting using the definition of projective limit. We omit further details.

Of particular interest are Weyl fractional derivatives of Dirichlet averages and their relation to the Dirichlet averages of Weyl fractional derivatives. To this end, let Ω again be an open convex subset of $\mathbb{R}^s$ or $\mathbb{C}^s$ and $f \in \mathcal{S}(\Omega)$. Let $z \in \mathbb{C}_>$ and let $n := \lceil \operatorname{Re} z \rceil$ and $\nu := n - z$. Furthermore, let $x := (x_1, \dots, x_s)^\top \in \Omega$. The Weyl partial fractional derivative ∂_i^z with respect to x_i , $i = 1, \dots, s$, of order z is defined by

$$\partial_i^z f(x) := \frac{\partial^z}{\partial x_i^z} f(x) := \frac{(-1)^n}{\Gamma(\nu)} \frac{\partial^n}{\partial x_i^n} \int_{\mathbb{R}_+^s} (t - x)^{\nu-1} f(t) dt.$$

Consider for a moment the case $s := 1$. Let $\zeta \in \bigtimes_{i=0}^k \Omega$ and denote by $\partial_i := \frac{\partial}{\partial \zeta_i}$, $i = 0, 1, \dots, k$, the partial derivative operator. Let $f : \Omega \rightarrow \mathbb{R}$ and consider the operator $\sum_{i=0}^k \partial_i = \langle \nabla, e(k) \rangle$, where ∇ is the k -dimensional

gradient and $e(k) = (1, \dots, 1) \in \mathbb{N}^k$. However, $\langle \nabla, e(k) \rangle f$ is equal to the *univariate* derivative $\frac{d}{dx} f(x, \dots, x)$, where $x := \zeta_1 = \dots = \zeta_k$. This suggests the following definition.

Definition 7. Let $g \in \mathcal{S}(\Omega)$ and let $z \in \mathbb{C}_{>}$ with $\lceil \operatorname{Re} z \rceil \leq k$. Then

$$\left(\sum_{i=0}^k \partial_i \right)^z g(\zeta) := \frac{(-1)^n}{\Gamma(\nu)} \frac{d^n}{dx^n} \int_{\mathbb{R}_+} t^{\nu-1} g(x+t, \dots, x+t) dt = \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_+} t^{\nu-1} g^{(n)}(x+t, \dots, x+t) dt.$$

Now, set $g(\zeta) := F(b(k); \zeta)$, with $b(k)$ a finite weight vector. Then, by the properties of Dirichlet averages and the fact that $\sum_{i=1}^k u_i = 1$, one obtains with $\partial(k)^z := \left(\sum_{i=0}^k \partial_i \right)^z$,

$$\begin{aligned} (\partial(k)^z F)(b(k); \zeta) &= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_+} t^{\nu-1} F^{(n)}(b(k); x+t, \dots, x+t) dt \\ &= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_+} t^{\nu-1} \frac{d^n}{dx^n} \int_{\Delta^k} f(u_1(x+t) + \dots + u_k(x+t)) d\mu_{b(k)}(u) \\ &= \frac{(-1)^n}{\Gamma(\nu)} \int_{\mathbb{R}_+} t^{\nu-1} \int_{\Delta^k} f^{(n)}(u_1(x+t) + \dots + u_k(x+t)) (u_1 + \dots + u_k)^k d\mu_{b(k)}(u) \\ &= \int_{\Delta^k} (\partial(k)^z f)(u \cdot \zeta) d\mu_{b(k)}(u). \end{aligned}$$

In quite similar fashion, one shows that

$$(\partial_i^z F)(b(k); \zeta) = \int_{\Delta^k} u_i^z f^{(z)}(u \cdot \zeta) d\mu_{b(k)}(u),$$

and, if $\{i_1, \dots, i_m\} \subseteq \{0, 1, \dots, k\}$, $m = 0, 1, \dots, k$,

$$(\partial_{i_1}^{z_{i_1}} \dots \partial_{i_m}^{z_{i_m}} F)(b(k); \zeta) = \int_{\Delta^k} u_{i_1}^{z_{i_1}} \dots u_{i_m}^{z_{i_m}} f^{(z_{i_1} + \dots + z_{i_m})}(u \cdot \zeta) d\mu_{b(k)}(u),$$

where $z_{i_\ell} \in \mathbb{C}_{>}$, $\ell = 1, \dots, m$, and $\lceil \operatorname{Re} z_{i_1} \rceil + \dots + \lceil \operatorname{Re} z_{i_m} \rceil \leq k$.

In order to extend the above results to Δ^∞ , we need to consider the operator $\partial := \partial(\infty) := \sum_{i=0}^\infty \partial_i$, where ∂_i denotes again the partial derivative with respect to ζ_i , $i \in \mathbb{N}_0$.

Remark 8. Operators of this type naturally act on functions $f : \mathbb{R}^\infty \rightarrow \mathbb{C}$ for which, for instance, the semi-norm $|f|_{1,\infty} := \sum_{i \in \mathbb{N}_0} \|\partial_i f\|_\infty < \infty$. In other words, the function f can be regarded as an element of the Sobolev space

$W^{1,\infty}(\mathbb{R}^\infty)$, defined as the projective limit of the Sobolev spaces $W^{1,\infty}(\mathbb{R}^n)$: $W^{1,\infty}(\mathbb{R}^\infty) := \varprojlim W^{1,\infty}(\mathbb{R}^n)$. In the current setting, however, these ideas will not be pursued further.

Instead, we consider the following scenario. Let $f \in \mathcal{S}(\mathbb{R}^\infty)$ and let $z \in \mathbb{C}_{>}$ with $n := \lceil \operatorname{Re} z \rceil$ and $\nu := n - z$. Define $\partial^z := \left(\sum_{i=0}^\infty \partial_i \right)^z$ to be an operator on $\mathcal{S}(\mathbb{R}^\infty)$ given by the expression

$$(\partial^z f)(\zeta) := \frac{(-1)^n}{\Gamma(\nu)} \frac{d^n}{dx^n} \int_0^\infty t^{\nu-1} f(x+t) dt = \frac{(-1)^n}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} f^{(n)}(x+t) dt,$$

where $\mathbb{R} \ni x := \zeta_1 = \zeta_2 = \dots = \zeta_n = \dots$. Replacing f by the Dirichlet average $G(b; \zeta)$ of some function $g \in \mathcal{S}(\mathbb{R}^\infty)$ and weight vector $b \in \ell^1(\mathbb{N}_0)$, one obtains by arguments similar to those given above that

$$(\partial^z G)(b; \zeta) = \int_{\Delta^\infty} g^{(z)}(u \cdot \zeta) d\mu_b(u),$$

and

$$(\partial_i^{z_i} G)(b; \zeta) = \int_{\Delta^\infty} u_i^{z_i} g^{(z_i)}(u \cdot \zeta) d\mu_b(u),$$

and also

$$(\partial_{i_1}^{z_{i_1}} \dots \partial_{i_m}^{z_{i_m}} G)(b; \zeta) = \int_{\Delta^\infty} u_{i_1}^{z_{i_1}} \dots u_{i_m}^{z_{i_m}} g^{(z_{i_1} + \dots + z_{i_m})}(u \cdot \zeta) d\mu_b(u),$$

for any $\{i_1, \dots, i_m\} \subseteq \mathbb{N}_0$.

Returning to the general case $s > 1$, the results obtained above apply to $\zeta \in \Omega^{\mathbb{N}_0} \subset (\mathbb{C}^s)^{\mathbb{N}_0}$ by considering partial derivatives with respect to the components ζ_i^j of $\zeta^j \in \zeta$. (See also Carlson¹² for the finite-dimensional vectorial setting.)

We will write again $\tau := \{t^n\}_{n \in \mathbb{N}_0}$ for a knot sequence in $\mathbb{R}^s$ satisfying the condition that $\limsup_{n \rightarrow \infty} \sqrt[n]{\|t^n\|} < \infty$. Furthermore we assume that the weight vector $b \in \ell^1(\mathbb{N}_0)$. Let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be a direction and let $z \in \mathbb{C}_>$. Employing Theorem 3 in Carlson¹² or Theorem 3.1 in Neuman & Van Fleet,⁸ to the functions $g^{(z)} \in \mathcal{D}(\mathbb{R}^\infty) \subset \mathcal{S}(\mathbb{R}^\infty)$ and $g_j^{(1+z)} := (\langle \lambda, t^j \rangle - \bullet)g^{(1+z)}$, $j \in \mathbb{N}_0$, yields for their Dirichlet averages on the knot sequence $\lambda\tau = \{\langle \lambda, t^n \rangle\}_{n \in \mathbb{N}}$

$$(c-1)G^{(z)}(b; \lambda\tau) = (c-1)G^{(z)}(b - e_j; \lambda\tau) + G_j^{(1+z)}(b; \lambda\tau), \quad (12)$$

where G_j is the Dirichlet average of g_j , and $c = \sum_{i \in \mathbb{N}_0} b_i$ as above, for a weight vector $b \in \ell^1(\mathbb{N}_0)$. Employing (7) to (12) we obtain

$$(c-1) \int_{\mathbb{R}} g^{(z)}(t) B_z(t | b; \lambda\tau) dt = (c-1) \int_{\mathbb{R}} g^{(z)}(t) B_z(t | b - e_j; \lambda\tau) dt + \int_{\mathbb{R}} (\langle \lambda, t^j \rangle - t) g^{(1+z)}(t) B_z(t | b; \lambda\tau) dt.$$

This last equation, however, is by the defining equation of multivariate complex B-splines $\mathbf{B}_z$ equivalent to

$$\begin{aligned} (c-1) \int_{\mathbb{R}^s} g^{(z)}(\langle \lambda, x \rangle) \mathbf{B}_z(x | b; \tau) dx &= (c-1) \int_{\mathbb{R}^s} g^{(z)}(\langle \lambda, x \rangle) \mathbf{B}_z(x | b - e_j; \tau) dx \\ &+ \int_{\mathbb{R}^s} \langle \lambda, t^j - x \rangle g^{(1+z)}(\langle \lambda, x \rangle) \mathbf{B}_z(x | b; \tau) dx, \quad j \in \mathbb{N}_0. \end{aligned} \quad (13)$$

We summarize these results in a theorem

Theorem 9. Let $\tau := \{t^n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}^s$ be a knot sequence with $\limsup_{n \rightarrow \infty} \sqrt[n]{\|t^n\|} < \infty$ and $b \in \ell^1 \mathbb{N}_0$ a weight vector. Assume that $\lambda \in \mathbb{R}^s \setminus \{0\}$ and $z \in \mathbb{C}_>$. Furthermore, assume that $g^{(z)} \in \mathcal{D}(\mathbb{R}^\infty)$ and let $g_j^{(1+z)} := (t^j - \bullet)g^{(1+z)}$, $j \in \mathbb{N}_0$. Then

$$\begin{aligned} (c-1) \int_{\mathbb{R}^s} g_\lambda^{(z)}(x) \mathbf{B}_z(x | b; \tau) dx &= (c-1) \int_{\mathbb{R}^s} g_\lambda^{(z)}(x) \mathbf{B}_z(x | b - e_j; \tau) dx \\ &+ \int_{\mathbb{R}^s} \langle \lambda, t^j - x \rangle g_\lambda^{(1+z)}(x) \mathbf{B}_z(x | b; \tau) dx, \quad j \in \mathbb{N}_0. \end{aligned}$$

Now suppose that $\tau = \{t^k\}_{k \in \mathbb{N}_0}$ is such that its convex hull $[\tau]$ does not contain $0 \in \mathbb{R}^s$, and let $n \in \mathbb{N}$. The R -geometric function $R_a(b; \tau) : \mathbb{R}_+^{n+1} \times \Omega^{n+1} \rightarrow \mathbb{C}$ is defined by

$$R_a(b; \tau) := \int_{\Delta^n} (\tau \cdot u)^a d\mu_b^n(u),$$

where $\Omega := H$, H a half-plane in $\mathbb{C} \setminus \{0\}$, if $a \in \mathbb{C} \setminus \mathbb{N}$, and $\Omega := \mathbb{C}$, if $a \in \mathbb{N}$. It can be shown (see Carlson¹¹) that R_{-a} , $a \in \mathbb{C}_>$, has a holomorphic continuation in τ to $\mathbb{C}_0$, where $\mathbb{C}_0 := \{\zeta \in \mathbb{C} \mid -\pi < \arg \zeta < \pi\}$.

Since this result holds for all $n \in \mathbb{N}$, the definition and properties of R_{-a} can be lifted to the infinite-dimensional simplex Δ^∞ using the properties of the projective limit, provided that the above-given conditions

on τ and the weight vector b are satisfied. Using (7), we can express R_{-a} as follows. Firstly, we require a result from Weyl fractional differentiation theory (see Section 2.2 in Kilbas et al.⁵), which states that for $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \geq 0$, and $\beta \in \mathbb{C}$

$$(W^{-\alpha} t^{\beta-1})(x) = \frac{\Gamma(1 + \alpha - \beta)}{\Gamma(1 - \beta)} x^{\beta-\alpha-1}, \quad (14)$$

provided that $\operatorname{Re}(\alpha + \beta - \lfloor \operatorname{Re} \alpha \rfloor) < 1$. Here $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, $x \mapsto \max\{n \in \mathbb{Z} | n \leq x\}$, denotes the *floor function*.

Suppose now that $z \in \mathbb{C}$ is such that $\operatorname{Re} z > 1$ and choose an $a \in \mathbb{C}_{>}$. Then, by virtue of (14), we can write

$$t^{-a} = \frac{\Gamma(a)}{\Gamma(a-z)} [t^{-(a-z)}]^{(z)},$$

provided $\operatorname{Re} a > 2 \operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor > 1$. Hence, with (7),

$$R_{-a}(b; \tau) = \int_{\Delta^\infty} (\tau \cdot u)^{-a} d\mu_b(u) = \frac{\Gamma(a)}{\Gamma(a-z)} \int_{\mathbb{R}} (t^{-(a-z)})^{(z)} B_z(t | b; \tau) dt, \quad (15)$$

for an $a \in \mathbb{C}$ satisfying $\operatorname{Re} a > 2 \operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$.

Assume that $\lambda \in \mathbb{R}^s \setminus \{0\}$ is a direction and that the knot sequence $\tau = \{t^k\}_{k \in \mathbb{N}_0}$ satisfies $\langle \lambda, t^k \rangle < 1$, for all $k \in \mathbb{N}_0$. Then,

$$\begin{aligned} R_{-a}(b; 1 - \lambda\tau) &= \frac{\Gamma(a)}{\Gamma(a-z)} \int_{\mathbb{R}} (t^{-(a-z)})^{(z)} B_z(t | b; 1 - \lambda\tau) dt = \frac{\Gamma(a)}{\Gamma(a-z)} \int_{\mathbb{R}} [(1-t)^{-(a-z)}]^{(z)} B_z(t | b; \lambda\tau) dt \\ &= \frac{\Gamma(a)}{\Gamma(a-z)} \int_{\mathbb{R}^s} [(1 - \langle \lambda, x \rangle)^{-(a-z)}]^{(z)} \mathbf{B}_z(x | b; \tau) dx. \end{aligned}$$

Hence, we proved the following theorem.

Theorem 10. Suppose that $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$ and $a \in \mathbb{C}$ are such that $\operatorname{Re} a > 2 \operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$. Moreover, let $\lambda \in \mathbb{R}^s \setminus \{0\}$ be such that $\langle \lambda, t^k \rangle < 1$, for all $k \in \mathbb{N}_0$. Then the R -geometric function R_{-a} can be expressed as

$$R_{-a}(b; 1 - \lambda\tau) = \frac{\Gamma(a)}{\Gamma(a-z)} \int_{\mathbb{R}^s} [K_{a-z, \lambda}(x)]^{(z)} \mathbf{B}_z(x | b; \tau) dx, \quad (16)$$

where $K_{a-z} := (1 - \bullet)^{-(a-z)}$.

Let us recall the following formula for the R -geometric function R_{-a} which is an extension of the finite-dimensional setting (see Theorem 6.8-3 in Carlson¹¹) to the infinite-dimensional case under the usual assumptions on the knots $\zeta := \{\zeta_n\}_{n \in \mathbb{N}_0}$, $\zeta_n > 0$ for all $n \in \mathbb{N}_0$, and the weight vector b .

$$R_{-a}(b; \zeta) = \prod_{n=0}^{\infty} \zeta_n^{-b_n} R_{a-c}(b; \zeta^{-1}), \quad (17)$$

where $c = \sum_{i=0}^{\infty} b_i$, with $c \notin -\mathbb{N}_0$, and $\zeta^{-1} := \{\zeta_n^{-1}\}_{n \in \mathbb{N}_0}$.

Now, choosing weights b_n , $n \in \mathbb{N}_0$, so that setting $a := c \in \mathbb{R}$ still satisfies the condition $a > 2 \operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor$, and using the fact that $R_0 = 1$, we obtain from (16) and (17),

$$\int_{\mathbb{R}^s} [K_{a-z, \lambda}(x)]^{(z)} \mathbf{B}_z(x | b; \tau) dx = \frac{\Gamma(a-z)}{\Gamma(a)} \prod_{n=0}^{\infty} (1 - \langle \lambda, t^n \rangle)^{-b_n},$$

which is a generalization of Watson's Identity. (Cf. Watson¹³ and Neuman & Van Fleet.⁸)

4. SUMMARY AND OUTLOOK

We extended the notion of complex B-spline to $\mathbb{R}^s$, $1 < s \in \mathbb{N}$, using an identity between univariate ordinary B-splines and multivariate ordinary B-splines involving ridge functions. In order to derive properties of these multivariate complex B-splines $\mathbf{B}_z$, we needed to employ Dirichlet averages generalized to infinite-dimensional simplices Δ^∞ . These generalized Dirichlet averages seem to play the same important role in multivariate complex B-spline theory as they do in the ordinary setting. We also derived an identity between multivariate complex B-splines and an R -geometric function involving Weyl fractional derivatives.

There are still several open problems and numerous directions along which research into the theory of multivariate complex B-splines may proceed. Some of these are as follows.

- What is the Fourier transform of multivariate complex B-splines and is it related to Dirichlet averages? (See Carlson¹² for such a connection between the S -geometric function and the Fourier transform of ordinary multivariate B-splines.)
- Is there an explicit representation of multivariate complex B-splines and if, are there connections to generalized complex difference and divided difference operators? (See also Forster & Massopust⁴ for $s = 1$.)
- How does one compute the moments of univariate and multivariate complex B-splines?
- Is it possible to construct scaling vectors and multi-scale representations from multivariate complex B-splines?
- Exhibit relations to the Radon transform.

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