On the Sturm comparison and convexity theorem for difference and q-difference equations

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Abstract

We give a Sturm-type comparison theorem and a convexity theorem for difference equations. We apply the convexity results to discrete orthogonal polynomials, such as the Hahn and Meixner polynomials by obtaining estimates on the second difference of their zeros. We show that the corresponding theorems for q-difference equations also hold, and present the results on the q-Laguerre polynomials.

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1 Introduction

Different forms of the Sturm comparison and convexity theorems play an important role in oscillation theory since they were first published by Sturm in the 1830's [12]. For example, the theory of oscillation for partial differential equations is mostly based on a generalized form of the Sturm comparison theorem. It is believed that Sturm first proved the comparison theorem for difference equations [2], nevertheless much of the related work done so far is on the continuous case for differential equations. There are analogues of the comparison and convexity theorems for difference equations [8] and a notable recent work to mention is a general form of the Sturm-Picone comparison theorem for higher order delay difference equations that appeared in [3].

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The Sturm convexity theorem (a consequence of the comparison theorem) for ordinary differential equations, first noted in [12], can be stated as follows.

Theorem. *(Sturm convexity theorem, cf. [1]) Let $y''(t) + F(t)y(t) = 0$ be a second-order differential equation in normal form, where F is continuous in (a, b). Let $y(t)$ be a nontrivial solution in (a, b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(t)$ in (a, b). Then*

1. *if $F(t)$ is strictly increasing in (a, b), then $x_{k+2} - x_{k+1} < x_{k+1} - x_k$,*
2. *if $F(t)$ is strictly decreasing in (a, b), then $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.*

In terms of the forward difference operator the first condition means that $\Delta^2 x_k = x_{k+2} - 2x_{k+1} + x_k < 0$ and the second condition means that $\Delta^2 x_k > 0$. Therefore, as an analogy, we say that the zeros are concave in the first case and convex in the second.

The goal of this paper is to find an analogue of these convexity results for solutions of difference and q-difference equations and apply them to orthogonal polynomials as it was done in the continuous case in e.g. [1, 5, 7, 13]. It is possible to carry over the classical proofs of the comparison and convexity theorems directly to difference equations of the type $\Delta^2 y(t-1) + F(t)y(t) = 0$, however this class of equations is too narrow. Note that any second order linear differential equation can be transformed into the normal form, while a similar transformation for general second order difference equations is – although possible – far too complicated, and it is impossible to determine the (monotonicity) properties of the resulting coefficient function $F(t)$. It is more productive to consider instead second order self-adjoint equations of the form

$$\Delta[p(t-1)\Delta y(t-1)] + q(t)y(t) = 0 \tag{1.1}$$

with $p(t) > 0$. As the following lemma shows, a solution of (1.1) can have only "simple" zeros.

Lemma. *(cf. [8, Lemma 6.1]) Let $y(t)$ be a nontrivial solution of (1.1) with $y(t_0) = 0$. Then $y(t_0 - 1)y(t_0 + 1) < 0$.*

For self-adjoint equations the discrete version of the Sturm separation theorem holds. It is a statement about generalized zeros (cf. [8, Section 6.2]).

Definition. *A solution $y(t)$ of (1.1) has a generalized zero at t_0 if either $y(t_0) = 0$ or $y(t_0 - 1)y(t_0) < 0$.*

Theorem. *(Sturm separation theorem) Two linearly independent solutions of (1.1) cannot have a common zero. If a nontrivial solution of (1.1) has a zero at t_1 and a generalized zero at $t_2 > t_1$, then any other linearly independent solution has a generalized zero in $(t_1, t_2]$. If a nontrivial solution of (1.1) has a generalized zero at t_1 and a generalized zero at $t_2 > t_1$, then any other linearly independent solution has a generalized zero in $[t_1, t_2]$.*

We will also need the notion of "disconjugacy". It was introduced by Philip Hartman in [4].

Definition. *Equation (1.1) is called disconjugate on* $[a, b]$ *if no nontrivial solution has two or more generalized zeros on* $[a, b]$*. Otherwise it is called conjugate.*

For the pair of equations

$$L_i y(t) = \Delta[p_i(t-1)\Delta y(t-1)] + q_i(t)y(t) = 0, \quad i = 1,\ 2$$

where $p_i(t) > 0$ in $[a, b+1]$, and $q_i(t)$ is defined on $[a+1, b+1]$, the statement below is often called the Sturm comparison theorem for difference equations. It is mostly used in oscillation theory.

Theorem. *(Sturm comparison theorem, cf. [8, Theorem 8.12]) Assume that* $q_1(t) \geq q_2(t)$ *on* $[a+1, b+1]$ *and* $p_2(t) \geq p_1(t) > 0$ *on* $[a, b+1]$*. If* $L_1 y(t) = 0$ *is disconjugate on* $[a, b+2]$*, then* $L_2 y(t) = 0$ *is disconjugate on* $[a, b+2]$*.*

In the following we give a new version of the Sturm comparison theorem, and then state as a consequence a new convexity theorem. They allow to obtain convexity results about the zeros of the solutions of self-adjoint difference equations, similar to the continuous case. In Section 2 we deal with difference equations and we apply the results to two families of discrete orthogonal polynomials, the Hahn and Meixner polynomials. In Section 3 we state and prove the corresponding results for q-difference equations and present the q-Laguerre polynomials as an example.

2 The comparison and convexity theorems for difference equations

Our main theorem is the following.

Theorem 1. *For the following pair of second order difference equations,*

$$\Delta[p_1(t-1)\Delta y(t-1)] + q_1(t)y(t) = 0, \tag{2.1}$$

$$\Delta[p_2(t-1)\Delta z(t-1)] + q_2(t)z(t) = 0, \tag{2.2}$$

assume that $p_2(t) \geq p_1(t) > 0$ *and* $q_1(t) \geq q_2(t)$ *on* $[t_0, t_0+n]$, $y(t_0) = z(t_0) = 0$, $y(t_0+1) > 0$ *and* $z(t_0+1) > 0$*. Suppose that* $y(t_0+2) > 0$, $y(t_0+3) > 0$, …, $y(t_0+n) > 0$ $(n \geq 2)$*. Then* $z(t_0+2) > 0$, $z(t_0+3) > 0$, …, $z(t_0+n) > 0$.

The theorem says that "$z(t)$ cannot change sign before $y(t)$ does".

Proof. Indirectly assume that $y(t_0+2) > 0$, $y(t_0+3) > 0$, …, $y(t_0+n) > 0$, and $z(t_0+2) > 0$, $z(t_0+3) > 0$, …, $z(t_0+k-1) > 0$, $z(t_0+k) < 0$, where $1 < k < n$. Then $z(t)$ has 2 generalized zeros on $[t_0,\ t_0+k]$. Therefore equation (2.2) is conjugate on $[t_0,\ t_0+k]$. According to the Sturm comparison theorem equation (2.1) is also conjugate on $[t_0, t_0+k]$, i.e. it has a nontrivial solution

$x(t)$ with at least two generalized zeros on $[t_0, t_0 + k]$. Since $k < n$, $x(t)$ and $y(t)$ must be linearly independent. By the Sturm separation theorem $x(t_0) \neq 0$. Hence $x(t)$ has two generalized zeros on $[t_0 + 1, t_0 + k]$. But then $y(t)$ must also have a generalized zero on $[t_0 + 1, t_0 + k]$, which is a contradiction. □

The following theorem could be considered as a possible version of the Sturm convexity theorem for difference equations.

Theorem 2. *Assume that, in equation (1.1) $p(t)$ is monotone decreasing, $q(t)$ is monotone increasing on $[t_0 - n, t_0 + n]$, $y(t_0) = 0$ and $y(t_0 + 1) > 0$. Then if $y(t_0 + 2) > 0$, $y(t_0 + 3) > 0$, ..., $y(t_0 + n) > 0$, then $y(t_0 - 1) < 0$, $y(t_0 - 2) < 0$, ..., $y(t_0 - n) < 0$.*

The result can be interpreted as the first sign change to the left of t_0 can not happen before the first sign change to the right of t_0. Note that the assumptions, according to the Lemma in the Introduction imply that $y(t_0 - 1) < 0$. Naturally, the analogue of the theorem with $p(t)$ increasing and $q(t)$ decreasing also holds.

Proof. Let $p_2(t) = p(2t_0 - t - 1)$ and $q_2(t) = q(2t_0 - t)$. It is easy to see that $z(t) := -y(2t_0 - t)$ satisfies equation (2.2). Now from the monotonicity assumptions it follows that on $[t_0, t_0 + n]$, $p_2(t) \geq p(t)$ and $q(t) \geq q_2(t)$. Therefore the statement follows from Theorem 1 above. □

Theorem 2 has a nice interpretation – in line with Sturm's original convexity theorem for differential equations – for continuous functions satisfying a self-adjoint difference equation, whose consecutive zeros are more than 1 unit apart. Although in this case we cannot conclude the actual convexity (or concavity) of zeros, we still have an estimate for the second difference.

Definition 2.1. *Let $y(t)$ be a continuous function on an interval (a, b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(t)$ in (a, b). Then*

- *if $\Delta^2 x_k < 1$ for all k, the zeros of $y(t)$ are called quasi-concave on (a, b),*
- *if $\Delta^2 x_k > -1$ for all k, the zeros of $y(t)$ are called quasi-convex on (a, b).*

Corollary 2.2. *Let $y(t)$ be a continuous function on an interval (a, b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(t)$ in (a, b). Assume that $\Delta x_k > 1$ for all k, and $y(t)$ satisfies equation (1.1) on (a, b).*

- *If $p(t)$ is monotone decreasing and $q(t)$ is monotone increasing on (a, b), then the zeros of $y(t)$ are quasi-concave on (a, b).*
- *If $p(t)$ is monotone increasing and $q(t)$ is monotone decreasing on (a, b), then the zeros of $y(t)$ are quasi-convex on (a, b).*

Proof. We only prove the first part, the proof of the second is analogue. Let x_{k+1} be a zero of $y(t)$ in (a, b). Assume that $y(x_{k+1} + 1) > 0$, the other case being analogous. Because of the assumption $\Delta x_k > 1$, $y(t)$ must be positive between x_{k+1} and $x_{k+1} + 1$ as well. Choose n so that $x_{k+1} + n < x_{k+2} \leq x_{k+1} + n + 1$.

Then $y(x_{k+1}+1)$, $y(x_{k+1}+2)$, ..., $y(x_{k+1}+n)$ are positive. Hence by Theorem 2, $y(x_{k+1}-1)$, $y(x_{k+1}-2)$, ..., $y(x_{k+1}-n)$ are negative. This, together with the assumption $\Delta x_k > 1$ and the Lemma in the Introduction means that $x_k < x_{k+1} - n$. On the other hand, by the choice of n, $x_{k+2} - x_{k+1} \leq n+1$. Putting these two inequalities together gives $\Delta^2 x_k < 1$. □

Example 1. *Hahn polynomials*
The Hahn polynomials $Q_n(x;\alpha,\beta,N)$ are defined as (cf. [6], [9])

$$Q_n(x;\alpha,\beta,N) = {}_3F_2(-n, n+\alpha+\beta+1, -x; \alpha+1, -N; 1) \quad n = 0,1,2,\ldots,N.$$

They are orthogonal for $\alpha > -1$ and $\beta > -1$ or $\alpha < -N$ and $\beta < -N$ with respect to a discrete weight, $w_1(x) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$ with masses at $x = 0,1,\ldots,N$. They have n simple zeros on $(0,N)$, any two consecutive zeros being more than one unit apart (cf. [10]). $y(x) := Q_n(x;\alpha,\beta,N)$ satisfies the difference equation

$$n(n+\alpha+\beta+1)y(x) = B(x)y(x+1) - [B(x)+D(x)]y(x) + D(x)y(x-1),$$

where $B(x) = (x+\alpha+1)(x-N)$ and $D(x) = x(x-\beta-N-1)$. Note that if $\alpha,\ \beta > -1$ and $0 < x < N$ then $B(x) < 0$ and $D(x) < 0$.

In order to apply the convexity theorem we have to bring the equation to self-adjoint form. To this end multiply both sides by $\prod_{s=0}^{x-1}(B(s)/D(s+1))$. Then we have the form (1.1) with

$$p(x) = \frac{-(\alpha+1)_{x+1}(-N)_{x+1}}{x!(-\beta-N)_x}$$

and

$$q(x) = n(n+\alpha+\beta+1)\frac{(\alpha+1)_x(-N)_x}{x!(-\beta-N)_x}.$$

Here $(\ )_n$ denotes Pochhammer's symbol defined by

$$\begin{aligned}(a)_n &= (a)(a+1).....(a+n-1) \text{ for } n \geq 1\\ (a)_0 &= 1 \text{ when } a \neq 0.\end{aligned}$$

In the following we assume that $\alpha+\beta > 0$. Set

$$t_1 := \frac{(\alpha+1)N}{\alpha+\beta+2} \quad \text{and} \quad t_2 := \frac{\alpha(N+1)}{\alpha+\beta}.$$

It is easy to see that $p(x)$ is monotone increasing $(p(x-1) < p(x))$ for $x < t_1$ and decreasing for $x > t_1$, and $q(x)$ is increasing for $x < t_2$ and decreasing for $x > t_2$. Therefore, between t_1 and t_2, p and q have opposing monotonicity, thus here Theorem 2 (or its analogue) is applicable. Note that with our assumptions $0 < t_1 < N$ always holds, while t_2 may also be negative or greater than N depending on the choice of α and β. Whether t_1 is greater than t_2 or not depends on the sign of $N(\beta-\alpha) - \alpha(\alpha+\beta+2)$. The possible cases are summarized below.

Proposition 2.3. *Let $\alpha+\beta>0$. The zeros of the Hahn polynomials $Q_n(x;\alpha,\beta,N)$ are quasi-convex or quasi-concave on certain intervals depending on the parameter values as follows.*

1. *If $-1<\alpha<0<\beta$ then the zeros are quasi-convex on $(0,t_1)$.*
2. *Let $0<\alpha<\beta$. If $N<\alpha/\beta$ then the zeros are quasi-concave on (t_1,N). If $\alpha/\beta<N<\alpha(\alpha+\beta+2)/(\beta-\alpha)$ then the zeros are quasi-concave on (t_1,t_2). If $N>\alpha(\alpha+\beta+2)/(\beta-\alpha)$ then the zeros are quasi-convex on (t_2,t_1).*
3. *If $-1<\beta<0<\alpha$ then the zeros are quasi-concave on (t_1,N).*
4. *Let $0<\beta<\alpha$. If $N<\alpha/\beta$ then the zeros are quasi-concave on (t_1,N). If $N>\alpha/\beta$ then the zeros are quasi-concave on (t_1,t_2).*

Example 2. *Meixner polynomials*
The Meixner polynomials $M_n(x;b,c)$ are defined as (cf. [6], [9])

$$M_n(x;b,c)={}_2F_1\left(-n,-x;b;1-\frac{1}{c}\right).$$

They are orthogonal for $b>0$ and $0<c<1$ with respect to the weight $w_2(x)=\frac{(\beta)_x}{x!}c^x$ with $x=0,1,\ldots$. It is possible to obtain (quasi-)convexity results for the zeros from difference equation, similar to the Hahn polynomials, however, it is easier to use the limit relation between the Hahn and Meixner polynomials. Taking $\alpha=b-1$, $\beta=N(1-c)/c$ in the definition of the Hahn polynomials and letting $N\to\infty$, we get the Meixner polynomials:

$$\lim_{N\to\infty}Q_n\left(x;b-1,N\frac{1-c}{c},N\right)=M_n(x;b,c).$$

With these substitutions the limit of t_1 is $bc/(1-c)$, the limit of t_2 is $(b-1)c/(1-c)$.

Proposition 2.4. *The zeros of the Meixner polynomials $M_n(x;b,c)$ are quasi-convex on*

1. $\left(0,\frac{bc}{1-c}\right)$ *if $b<1$,*
2. $\left(\frac{(b-1)c}{1-c},\frac{bc}{1-c}\right)$ *if $b\geq 1$.*

The first case follows from Case 1. of Proposition 2.3 and the second case follows from the last sub-case of Case 2. of Proposition 2.3.

3 The comparison and convexity theorems for q-difference equations

The results in Section 2 can also be formulated for q-difference equations. The q-difference operator is defined by ($0 < q < 1$):

$$(Df)(x) = \frac{f(x) - f(qx)}{x(1-q)}. \tag{3.1}$$

Theorem 3. *For the following pair of second order q-difference equations,*

$$D(h_1 Dy)(q^{-1}x) + f_1(x)y(x) = 0 \tag{3.2}$$

$$D(h_2 Dz)(q^{-1}x) + f_2(x)z(x) = 0 \tag{3.3}$$

assume that $h_2(q^t x_0) \geq h_1(q^t x_0) > 0$ and $f_1(q^t x_0) \geq f_2(q^t x_0)$ for $t = 0, 1, \ldots, n$, $y(x_0) = z(x_0) = 0$, $y(qx_0) > 0$ and $z(qx_0) > 0$. Suppose that $y(q^2 x_0) > 0$, $y(q^3 x_0) > 0, \ldots, y(q^n x_0) > 0$. Then $z(q^2 x_0) > 0$, $z(q^3 x_0) > 0, \ldots, z(q^n x_0) > 0$.

Proof. We can transform equations (3.2) and (3.3) into difference equations. With the substitution $x = q^t x_0$ and $Y(t) := y(q^t x_0)$ equation (3.2) becomes

$$qh_1(q^t x_0)Y(t+1) + [(1-q)^2 q^{2t} x_0^2 f_1(q^t x_0) - q^2 h_1(q^{t-1}x_0) - qh_1(q^t x_0)]Y(t) \\ + q^2 h_1(q^{t-1}x_0)Y(t-1) = 0.$$

Since $h_1 > 0$, this equation is self-adjoint, and just like in Example 1, it can be written in the standard form (1.1). For this multiply both sides by q^{-t}. Then we have equation (2.1) for $Y(t)$ with $p_1(t) = q^{1-t}h_1(q^t x_0)$ and $q_1(t) = (1-q)^2 q^t x_0^2 f_1(q^t x_0)$. We can transform (3.3) similarly and then the statement of Theorem 3 follows from Theorem 1. □

We can also formulate the q-version of the convexity theorem. For that we need the q-version of [8, Lemma 6.1], saying, basically, that a nontrivial solution of a second order self-adjoint q-difference equation cannot have a "double" zero.

Lemma 3.1. *Assume that $y(x)$ is a nontrivial solution of*

$$D(hDy)(q^{-1}x) + f(x)y(x) = 0. \tag{3.4}$$

Assume that $h(x) > 0$ and $y(x_0) = 0$. Then $y(q^{-1}x_0)y(qx_0) < 0$.

Proof. Since $y(x_0) = 0$, we have from equation (3.4)

$$qh(q^{-1}x_0)y(q^{-1}x_0) + h(x_0)y(qx_0) = 0.$$

If $y(q^{-1}x_0) = 0$ or $y(qx_0) = 0$, then y would be trivial. Hence the statement follows from the positivity of h. □

In the following we assume that $x_0 > 0$.

Theorem 4. *Let $y(x)$ be a solution of (3.4) with $h(x)/x$ monotone increasing and $xf(x)$ monotone decreasing on the set $\{x = q^t x_0 \mid t = -n,\ -n-1, \dots, n\}$. Assume that $y(x_0) = 0$ and $y(qx_0) > 0$. Then if $y(q^2x_0) > 0$, $y(q^3x_0) > 0$, ..., $y(q^nx_0) > 0$ then $y(q^{-1}x_0) < 0$, $y(q^{-2}x_0) < 0$, ..., $y(q^{-n}x_0) < 0$.*

Note that Lemma 3.1 implies that $y(q^{-1}x_0) < 0$. The analogues of the theorem with $x_0 < 0$ or the monotonicities reversed also hold.

Proof. Set

$$h_2(x) = q\left(\frac{x}{x_0}\right)^2 h\left(q^{-1}\frac{x_0^2}{x}\right) \quad \text{and} \quad f_2(x) = \left(\frac{x_0}{x}\right)^2 f\left(\frac{x_0^2}{x}\right).$$

Then $z(x) := -y(x_0^2/x)$ satisfies equation (3.3). It is easy to see that the monotonicity assumptions on h and f guarantee that $h_2(x) \geq h(x)$ and $f(x) \geq f_2(x)$ for $x = q^t x_0$, $t = 0, 1, \dots, n$. Hence the statement follows from Theorem 3. □

We can also interpret this q-convexity theorem for continuous functions like we did with Theorem 2. Now instead of the distance of the zeros, what matters is the quotient on a q-logarithmic scale.

Definition 3.2. *(cf. [11]) Let $y(x)$ be continuous on an interval (a, b) with $0 < a < b$. We say that the zeros of $y(x)$ are well separated if $y(c) = y(d) = 0$ and $c < d$ implies that $d/c > q^{-1}$.*

Definition 3.3. *Let $y(x)$ be a continuous function on an interval (a, b) with $0 < a < b$, and let $x_1 < \dots < x_k < x_{k+1} < \dots$ denote the consecutive zeros of $y(x)$ in (a, b). Then*

- *if $x_k/x_{k-1} > qx_{k+1}/x_k$ for all k, the zeros of $y(x)$ are called q-quasi-concave on (a, b),*
- *if $x_{k+1}/x_k > qx_k/x_{k-1}$ for all k, the zeros of $y(x)$ are called q-quasi-convex on (a, b).*

Remark 3.4. *$y(x)$ is q-quasi-concave (convex) if and only if $y(q^{-x})$ is quasi-concave (convex).*

With these notions we have the following consequence of Theorem 4. It can be proved the same way from Theorem 4 and Lemma 3.1 as Corollary 2.2 from Theorem 2, therefore the proof is omitted.

Corollary 3.5. *Let $y(x)$ be a continuous function on an interval (a, b) with $0 < a < b$, and let $x_1 < \dots < x_k < x_{k+1} < \dots$ denote the consecutive zeros of $y(x)$ in (a, b). Assume that the zeros of y are well separated and $y(x)$ satisfies equation (3.4) on (a, b).*

- *If $h(x)/x$ is monotone decreasing and $xf(x)$ is monotone increasing on (a, b), then the zeros of $y(x)$ are q-quasi-concave on (a, b).*

- *If $h(x)/x$ is monotone increasing and $xf(x)$ is monotone decreasing on (a,b), then the zeros of $y(x)$ are q-quasi-convex on (a,b).*

Example 3. *q-Laguerre polynomials*
The q-Laguerre polynomials are defined through a basic hypergeometric function (cf. [9]):

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\ {}_1\phi_1\left(q^{-n}, q^{\alpha+1}; q; -q^{n+\alpha+1}x\right)$$

For the definition of basic hypergeometric functions or q-shifted factorials see [6] or [9]. The q-Laguerre polynomials are orthogonal for $\alpha > -1$ on $(0,\infty)$ with respect to a discrete weight with masses at cq^k, $k \in \mathbb{Z}$, where c is any positive number. From the orthogonality relation follows that they have n simple positive zeros and they are well-separated (cf. [11]). $y(x) := L_n^{(\alpha)}(x;q)$ satisfies the q-difference equation

$$-q^\alpha(1-q^n)xy(x) = q^\alpha(1+x)y(qx) - [1+q^\alpha(1+x)]y(x) + y(q^{-1}x).$$

Again, to apply the convexity results, we have to transform this equation in the form (3.4). It can be achieved by multiplying both sides by $x^{\alpha+1}/(-x;q)_\infty$. A short calculation shows that then we have (3.4) with

$$h(x) = \frac{q^\alpha x^{\alpha+1}}{(-qx;q)_\infty} \quad \text{and} \quad f(x) = \frac{q^{\alpha-1}(1-q^n)x^\alpha}{(1-q)^2(-x;q)_\infty}.$$

Now $h(x)/x$ is a nonnegative function with $\lim_{x\to\infty} h(x)/x = 0$. For $-1 < \alpha \le 0$ it is monotone decreasing on $(0,\infty)$. For $\alpha > 0$, on the other hand $\lim_{x\to 0} h(x)/x = 0$. Setting the logarithmic derivative equal to 0 gives the equation

$$\sum_{i=1}^{\infty} \frac{q^i x}{1+q^i x} = \alpha. \tag{3.5}$$

The left hand side of (3.5) is strictly monotone increasing from 0 to ∞ on $(0,\infty)$, therefore there is exactly one positive solution, $x = t_0(\alpha)$, which is strictly increasing in α. Hence for $\alpha > 0$ the function $h(x)/x$ is monotone increasing on $(0, t_0(\alpha))$ and decreasing on $(t_0(\alpha), \infty)$. Similarly, the logarithmic derivative of $xf(x)$ is 0 when

$$\sum_{i=0}^{\infty} \frac{q^i x}{1+q^i x} = \alpha + 1. \tag{3.6}$$

Thus there is also exactly one $x = s_0(\alpha) > 0$ solving this equation, so that $xf(x)$ is monotone increasing on $(0, s_0(\alpha))$ and decreasing on $(s_0(\alpha), \infty)$. Equation (3.6) can also be written in the form

$$-\frac{1}{1+x} + \sum_{i=1}^{\infty} \frac{q^i x}{1+q^i x} = \alpha,$$

which, in comparison with (3.5) shows that $t_0(\alpha) < s_0(\alpha)$. All this together with Corollary 3.5 gives the following result.

Proposition 3.6. *With the above notation the zeros of the q-Laguerre polynomial* $L_n^{(\alpha)}(x;q)$ *are q-quasi-concave on*

- $(0, s_0(\alpha))$ *if* $-1 < \alpha \leq 0$,
- $(t_0(\alpha), s_0(\alpha))$ *if* $\alpha > 0$.

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