Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 266 (2019) 1–43
Journal of Differential Equations
www.elsevier.com/locate/jde

Unidirectional evolution equations of diffusion type

Goro Akagi [a,b,c,*], Masato Kimura [d]

[a] *Mathematical Institute, Tohoku University, 6-3 Aoba, Aramaki, Aoba-ku, Sendai 980-8578, Japan*
[b] *Helmholtz Zentrum München, Institut für Computational Biology, Ingolstädter Landstraße 1, 85764 Neuherberg, Germany*
[c] *Technische Universität München, Zentrum Mathematik, Boltzmannstraße 3, D-85748 Garching bei München, Germany*
[d] *Faculty of Mathematics and Physics, Kanazawa University, Kakuma, Kanazawa 920-1192, Japan*

Received 23 September 2017; revised 7 February 2018
Available online 29 October 2018

Abstract

This paper is concerned with the uniqueness, existence, *partial* smoothing effect, comparison principle and long-time behavior of solutions to the initial-boundary value problem for a *unidirectional diffusion equation*. The unidirectional evolution often appears in Damage Mechanics due to the strong irreversibility of crack propagation or damage evolution. The existence of solutions is proved in an L^2-framework by employing a backward Euler scheme and by introducing a new method of a priori estimates based on a reduction of discretized equations to variational inequalities of obstacle type and by developing a regularity theory for such obstacle problems. The novel discretization argument will be also applied to prove the comparison principle as well as to investigate the long-time behavior of solutions.

MSC: primary 35K86; secondary 35K61, 74A45

Keywords: Unidirectional diffusion equation; Damage mechanics; Discretization; Variational inequality of obstacle type; Regularity; Subdifferential calculus

* Corresponding author.
E-mail addresses: goro.akagi@tohoku.ac.jp (G. Akagi), mkimura@se.kanazawa-u.ac.jp (M. Kimura).

https://doi.org/10.1016/j.jde.2018.05.022

Contents

1. Introduction and main results

1.1. Introduction

Dynamics of various phase transition phenomena are described by phase-field approaches, where a phase (or order) parameter is introduced to specify the state of phase and their dynamics are often described in terms of gradient flows of appropriate free energy functionals (e.g., Cahn–Hilliard and Allen–Cahn equations). Phase-field approaches are also applied to Damage Mechanics. On the other hand, by reflecting a significant feature of damage phenomena (e.g., crack propagation and damage accumulation), evolution of damage is constrained to be *unidirectional*. Indeed, crack propagation is an irreversible phenomena, and particularly, cracks in a specimen or the damage of a material (e.g., microcracks which break or weaken bonds of microstructures) never autonomously disappear nor decrease. Therefore evolution of phase parameters (e.g. damage variable) are usually supposed to be unidirectional, i.e., nondecreasing or nonincreasing (see, e.g., [20,28,36,38,39,46,57,56,35,13,14]). Such unidirectional evolution processes are often described by gradient systems (in PDE forms) involving the *positive-part function*, $s \mapsto (s)_+ := s \vee 0 = \max\{s, 0\}$ for $s \in \mathbb{R}$.

In order to find out mathematical features of such gradient systems with unidirectional constraint, in this paper, we shall treat, as a simplest case, the evolution of $u = u(x,t)$ governed by the following fully nonlinear PDE,

$$\partial_t u = \left(\Delta u + f\right)_+, \qquad \text{for } x \in \Omega,\ t > 0, \tag{1}$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^n$ with $n \in \mathbb{N}$, $\partial_t u = \partial u/\partial t$, Δ stands for the n-dimensional Laplacian and $f = f(x,t)$ is a given function. More precisely, the main purpose of this paper is to prove the uniqueness, existence, *partial* smoothing effect and comparison principle of *strong* solutions $u = u(x,t)$ of the initial-boundary value problem for (1) and to reveal the asymptotic behavior of $u = u(x,t)$ as $t \to \infty$.

Solutions of (1) exhibit *unidirectional* nature, more precisely, the non-decrease of $u = u(x,t)$ in t, since the right-hand side of (1) is non-negative due to the presence of the positive part function. Such unidirectional evolutions appear in various fields of natural sciences and engineering fields. Here let us briefly discuss a couple of complete models which arise from Damage Mechanics and involve the unidirectional constraint, although it may be beyond the scope of the present paper. In [57] (see also [56]), Takaishi and Kimura proposed the following phase field model of crack propagation which contains a unidirectional evolution equation of diffusion type:

$$\alpha_1 w_t = \mu \mathrm{div}\left((1-u)^2 \nabla w\right) \quad \text{in } \Omega \times (0,\infty), \tag{2}$$

$$\alpha_2 u_t = \left(\varepsilon \mathrm{div}(\gamma(x)\nabla u) - \frac{\gamma(x)}{\varepsilon} u + \mu |\nabla w|^2 (1-u)\right)_+ \quad \text{in } \Omega \times (0,\infty), \tag{3}$$

where $w = w(x,t)$, $u = u(x,t)$, $\mu > 0$, $\gamma(x)$, and $\varepsilon > 0$ denote a scalar anti-plane displacement of a two dimensional elastic plate $\Omega \subset \mathbb{R}^2$, a damage variable, a shear modulus, a fracture toughness, and a regularization parameter, respectively, and $\alpha_1 \geq 0$, $\alpha_2 > 0$ are given constants. More precisely, in view of numerical analysis, they introduced a phase parameter (or damage variable) $u = u(x,t) \in [0,1]$ describing the crack configuration (e.g., $u \approx 1$ and $u \approx 0$ mean "totally cracked" and "not cracked" states, respectively). Since crack propagation is a unidirectional phenomenon and $u(x,t)$ is thereby supposed to be nondecreasing, the evolution of the damage variable u is often described by means of an evolution equation with the positive-part function such as (3). Several numerical computations for the crack propagation model stated above are shown in [57].

The mathematical model (2) and (3) is derived as a gradient flow of the following energy:

$$\mathcal{F}_\varepsilon(w,u) = \frac{\mu}{2} \int_\Omega (1-u)^2 |\nabla z|^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega \gamma(x) \left(\varepsilon |\nabla u|^2 + \frac{u^2}{\varepsilon}\right) \mathrm{d}x.$$

This is a kind of Ambrosio–Tortorelli regularization [6,7] of the so-called *Francfort–Marigo energy*, which describes a quasi-static evolution of brittle fractures in elastic bodies based on *Griffith's criterion* (see [25] for more details). This type of regularized energy is also considered in the context of the minimizing movement model for crack propagation [28] and the convergence of the regularized minimizing movement to a quasi-static brittle fracture model as $\varepsilon \to 0$ is proved there. The constant α_1 in (2) is often chosen as $\alpha_1 = 0$, which corresponds to the quasi-static crack propagation. We refer the readers to [36,57] for the case $\alpha_1 > 0$ (see also [20] for a minimizing movement version). Recently, rate-independence of evolution of u has been also taken into account (see [38–40] and references therein). To the best of authors' knowledge, existence of solutions to the full system (2), (3) above has not yet been fully pursued, although some modified models have already been well studied in several directions (see, e.g., [15,16,50]). Equation (1) can be regarded as a simplified one of (3), and the theory which will be established in the present paper may shed new light on studies on the solvability for more complicated equations including (2), (3) with the aid of fixed point arguments. In such a point of view, it is also meaningful to consider (1) under assumptions for f as weak as possible.

In mathematical points of view, (1) is classified as a fully nonlinear PDE, which is not fit for energy methods in general; however, by employing a (multi-valued) inverse function of the positive part function $(\cdot)_+$, (1) can be formulated as a sort of doubly nonlinear evolution equations,

$$\partial_t u + \alpha(\partial_t u) - \Delta u \ni f \quad \text{a.e. in } \Omega \times (0, \infty), \tag{4}$$

where α is a (multi-valued) maximal monotone function in $\mathbb{R}$ given by $\alpha(0) = (-\infty, 0]$ and $\alpha(s) = \{0\}$ for any $s > 0$ with the domain $D(\alpha) = [0, \infty)$ (see Section 3 below for more details). Equation (4) is fitter for energy methods and monotone techniques. On the other hand, in view of the L^2-theory of evolution equations, two operators $v \mapsto \alpha(v(\cdot))$ and $v \mapsto -\Delta v$ (defined for $v \in L^2(\Omega)$) are unbounded in $L^2(\Omega)$, and hence, it is more delicate to establish a priori estimates for proving the existence of strong solutions, as compared with standard equations without unidirectional constraints, e.g., the classical and nonlinear diffusion equations.

The nonlinear PDE (4) may fall within the frame of abstract doubly nonlinear evolution equations in a Hilbert space H of the form

$$\partial\Psi(\partial_t u(t)) + \partial\Phi(u(t)) \ni f(t) \quad \text{in } H, \quad 0 < t < T, \tag{5}$$

where $\partial\Psi$ and $\partial\Phi$ denote the subdifferential operator of functionals $\Psi : H \to (-\infty, \infty]$ and $\Phi : H \to (-\infty, \infty]$, respectively. In a thermodynamic approach to continuum mechanics, Ψ and Φ are often referred to as a *dissipation functional* and an *energy functional*, respectively. To reduce (4) into the form (5), we set $u(t) := u(\cdot, t)$ and particularly choose

$$H = L^2(\Omega), \quad \Psi(v) = \frac{1}{2}\int_\Omega |v|^2 \, dx + I_{[\,\cdot\, \geq 0]}(v), \quad \Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx \quad \text{for } v \in H,$$

where $I_{[\,\cdot\,\geq 0]}$ is the indicator function over the set $\{v \in L^2(\Omega) : v \geq 0 \ \text{a.e. in } \Omega\}$. Then we note that both subdifferentials $\partial\Psi$ and $\partial\Phi$ are unbounded in H. Let us briefly review the previous studies on abstract doubly nonlinear evolution equations such as (5). Barbu [12] proved the existence of solutions for (5) with two unbounded operators $\partial\Psi$ and $\partial\Phi$ by using the elliptic-in-time regularization and by imposing the differentiability (in t) of f. This result was generalized by Arai [8], Senba [54] and so on. In these papers, the term $\partial\Psi(\partial_t u(t))$ is estimated by differentiating the equation and by testing it with $\partial\Psi(\partial_t u(t))$. Therefore the differentiability of f and regularity of initial data (more precisely, $f \in W^{1,1}(0, T; H)$ and $u_0 \in D(\partial\Phi)$ in [8]) are essentially required, and moreover, some strong monotonicity condition (i.e., the so-called $\partial\Psi$-monotonicity) is also imposed on $\partial\Phi$. Similar methods of establishing a priori estimates are also used in individual studies on irreversible phase transition models (see, e.g., [17]). On the other hand, Colli and Visintin established an alternative approach to (5) in [23], where $\partial\Psi$ is supposed to be bounded and coercive with linear growth instead of assuming the regularity assumption on f and the $\partial\Psi$-monotonicity of $\partial\Phi$ (see also [22]). Their framework would be more flexible in view of applications to nonlinear PDEs and has been extensively applied to various types of doubly nonlinear problems. Moreover, their framework has been generalized in various directions, e.g., perturbation problems, long-time behaviors (see, e.g., [1,49,48], [51, Sect. 11], [55,53,47,52,2–5]). However, due to the unboundedness of $\partial\Psi$, (4) seems to be beyond the scope of the latter approach. On the other hand, the former approach due to Barbu and Arai is applicable to (4), provided that $f \in W^{1,1}(0, T; L^2(\Omega))$ and $u_0 \in D(\partial\Phi) \subset H^2(\Omega)$. Aso et al. [9,10] also treated an irreversible phase transition system in a different fashion (see also [26]); however, the regularity conditions $f \in W^{1,2}(0, T; H)$ and $u_0 \in D(\partial\Phi)$ are also assumed there. As for (1) with $f \equiv 0$, one can also find some result in [37]. Moreover, in [29,30], some abstract framework is developed based on the so-called Minimizing Movement and it is also applicable.

We further refer the reader to a recent paper [45], where a waiting time effect is studied for the one-dimensional equation on the real line.

In this paper, we present a novel approach to (1) (or equivalently (4)) by introducing a reformulation of discretized equations for (1) by means of elliptic variational inequalities of obstacle type. Moreover, by developing a regularity theory of such elliptic obstacle problems, we shall establish new a priori estimates for (1) (or (4)) and prove "partial" smoothing effect of solutions without assuming the differentiability (in t) of f. Such a relaxation of the regularity assumption on f may bring some advantage to develop a perturbation theory (towards, e.g., a crack propagation model in [57]). We also stress that, to our knowledge, no smoothing effect has been proved so far for doubly nonlinear evolution equations of type (5). Moreover, the novel discretization argument will be also applied to investigate the long-time behavior of solutions as well as to prove a comparison theorem. In particular, we shall provide uniform (in t) estimates for solutions by employing the method of a priori estimates developed in the existence part. Furthermore, some variational inequality of obstacle type will play a crucial role in asymptotic analysis; indeed, it will turn out that every solution will converge to the unique solution $z = z(x)$ of a variational inequality of obstacle type involving the initial data as an obstacle function from below under suitable assumptions. Here it is worth mentioning that the limit z of the solution $u = u(x,t)$ depends on its initial data u_0; indeed, one can construct different limits of solutions for different initial data and they may be accumulating. On the other hand, the ω-limit set of each solution is a singleton.

1.2. Problem and main results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \in \mathbb{N}$. Let Γ be the boundary of Ω and let Γ_{D} and Γ_{N} be (relatively) open subsets of Γ such that

$$\mathcal{H}^{n-1}(\Gamma \setminus (\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}})) = 0, \quad \Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset,$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. One of these two subsets may be empty. In such a case, the other set coincides with the whole of Γ. Let ν denote the outward-pointing unit normal vector on Γ. Main results of the present paper are concerned with the following initial-boundary value problem for a unidirectional evolution equation of diffusion type,

$$\partial_t u = \left(\Delta u + f\right)_+ \quad \text{in } Q := \Omega \times (0,\infty), \tag{6}$$

$$u = 0 \quad \text{on } \Gamma_{\mathrm{D}} \times (0,\infty), \tag{7}$$

$$\partial_\nu u = 0 \quad \text{on } \Gamma_{\mathrm{N}} \times (0,\infty), \tag{8}$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{9}$$

where $\partial_t = \partial/\partial t$, $f = f(x,t)$ and $u_0 = u_0(x)$ are given functions of class $L^2_{loc}([0,\infty); L^2(\Omega))$ and $L^2(\Omega)$, respectively, and $\partial_\nu u := \nabla u \cdot \nu$ denotes the normal derivative of u. Moreover, $(\cdot)_+$ stands for the positive part function, i.e., $(s)_+ := s \vee 0$ for $s \in \mathbb{R}$. If Γ_{D} (resp., Γ_{N}) is empty, the corresponding boundary condition (7) (resp., (8)) is ignored.

Remark 1.1. By change of variable, another unidirectional diffusion equation,

$$\partial_t u = -\big(\Delta u + f\big)_- \quad \text{in } Q, \tag{10}$$

where $(s)_- := (-s) \wedge 0 \geq 0$ for $s \in \mathbb{R}$, is reduced to (6). Indeed, set $v := -u$ and $g := -f$. Then (10) is transformed to

$$-\partial_t v = -\big(-\Delta v - g\big)_- = -\big(\Delta v + g\big)_+,$$

whence v solves (6) with f replaced by g.

Remark 1.2. In this paper we focus on homogeneous boundary conditions, since we study (1) as a simplified equation of (3), which is an evolution equation for phase parameter, and then, the homogeneous Neumann and Dirichlet conditions seem appropriate from a physical point of view. From a mathematical point of view, of course, one can extend the results of the paper for inhomogeneous ones, provided inhomogeneous boundary data can be absorbed by the external force $f(x,t)$ and initial data $u_0(x)$. For instance, replace (7) by $u = \rho$ on $\Gamma_D \times (0,\infty)$ with a datum $\rho : \partial\Omega \to \mathbb{R}$ which can be extended to a function $\bar{\rho}$ in $H^2(\Omega)$ satisfying $\partial_\nu \rho = 0$ on Γ_N. Then set $\tilde{u} := u - \bar{\rho}$, $\tilde{u}_0 := u_0 - \bar{\rho}$ and $\tilde{f} := f + \Delta\bar{\rho}$ and reduce the problem to a homogeneous boundary problem (6)–(9) with u, f and u_0 replaced by $\tilde{u}$, $\tilde{f}$ and $\tilde{u}_0$, respectively. Otherwise, we may need to modify the regularity theory to be developed in §2, which is not straightforward.

Let us start with defining *strong solutions* of (6)–(9). To this end, we set up notation. Let $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ denote the trace operator defined on $H^1(\Omega)$ (throughout the paper, we may omit γ_0 if no confusion can arise). Here and henceforth, $B(U,W)$ stands for set of all bounded linear operators from U to W. Moreover, define

$$V := \{v \in H^1(\Omega) : \ \gamma_0 v = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_{\mathrm{D}}\},$$
$$X := \{v \in H^2(\Omega) : \ \gamma_0(\nabla v) \cdot \nu = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_{\mathrm{N}}\},$$

equipped with the induced norms and inner products, i.e., $\|\cdot\|_V = \|\cdot\|_{H^1(\Omega)}$ and $(\cdot,\cdot)_V = (\cdot,\cdot)_{H^1(\Omega)}$ for V; $\|\cdot\|_X = \|\cdot\|_{H^2(\Omega)}$ and $(\cdot,\cdot)_X = (\cdot,\cdot)_{H^2(\Omega)}$ for X. Then V and X are closed subspaces of $H^1(\Omega)$ and $H^2(\Omega)$, respectively; hence, they are Hilbert spaces. We denote by $\langle\cdot,\cdot\rangle_V$ the duality pairing between V and its dual space V'. If either Γ_{D} or Γ_{N} is empty, the corresponding boundary condition specified in the definition of V or X above is ignored. Furthermore, we assume throughout the paper that $C(\overline{\Omega}) \cap V$ is dense in V.

Remark 1.3. The density of $C(\overline{\Omega}) \cap V$ in V can be checked for smooth domains as well as for Lipschitz domains with Γ_D satisfying appropriate conditions (see [43, Lemma 5.3] and references therein). On the other hand, it will be used only in the proof of Theorem 2.4 for the case where the obstacle function ψ is not sufficiently smooth (see Remark 2.7). Hence, if u_0 lies on $H^2(\Omega)$, the density is not necessary for the main results stated in this subsection.

We are concerned with *strong solutions* of (6)–(9) defined by

Definition 1.4 *(Strong solution).* For $T > 0$, a function $u \in C([0,T]; L^2(\Omega))$ is called a *strong solution* of (6)–(9) on $[0,T]$, if the following three conditions are satisfied:

(i) $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; X \cap V)$,
(ii) the equation $\partial_t u = (\Delta u + f)_+$ holds a.e. in $Q_T := \Omega \times (0, T)$,
(iii) the initial condition $u|_{t=0} = u_0$ holds a.e. in Ω.

A function $u \in C([0, \infty); L^2(\Omega))$ is called a *strong solution* of (6)–(9) on $[0, \infty)$, if for any $T > 0$, the restriction of u onto $[0, T]$ is a strong solution of (6)–(9) on $[0, T]$.

Remark 1.5. One can further derive that $u \in C([0, T]; V)$ from (i) by employing a chain-rule for convex functionals. See Lemma 3.4 below for more details.

We are now in position to state main results, whose proofs will be given in later sections. We begin with the uniqueness of solutions.

Theorem 1.6 *(Uniqueness). Let $T > 0$, $u_0 \in V$ and $f \in L^2(Q_T)$. Then the strong solution of* (6)–(9) *on* $[0, T]$ *is unique.*

To state our existence result, we shall introduce some assumptions for the domain Ω and the boundary Γ_D, Γ_N. For $\lambda \in \mathbb{R}$, we define a mapping $A_\lambda \in B(V, V')$ by

$$\langle A_\lambda u, v\rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv)\,\mathrm{d}x \quad \text{for } u, v \in V, \tag{11}$$

that is, $A_\lambda = -\Delta + \lambda$ in a weak formulation. It is well known that $A_\lambda \in \mathrm{Isom}(V, V')$ holds if $\lambda > 0$ (here $\mathrm{Isom}(V, V')$ means the set of all bijective operators in $B(V, V')$). Hence one can define $u = A_\lambda^{-1} g$ for $g \in L^2(\Omega)$ as the unique solution u of the elliptic problem in a weak form,

$$\int_\Omega (\nabla u \cdot \nabla v + \lambda uv)\,\mathrm{d}x = \int_\Omega gv\,\mathrm{d}x \quad \text{for all } v \in V,$$

(i.e., $A_\lambda u = g$ in V'). Then we assume that

$$A_1^{-1} g \in H^2(\Omega) \quad \text{for all } g \in L^2(\Omega). \tag{12}$$

Condition (12) is often called an *elliptic regularity* condition and deeply related to the geometry of the domain and boundary conditions. Indeed, it holds true for smooth domains with a single boundary condition (i.e., $\Gamma_N = \emptyset$ or $\Gamma_D = \emptyset$). However, it is more delicate to consider the validity of (12) for situations with nonsmooth domains or mixed boundary conditions. On the other hand, in order to take account of physical backgrounds of crack growth models and their numerical simulations, the regularity of the boundary may be at most Lipschitz continuous, and mixed boundary conditions seem to be natural as well. We shall give conditions equivalent to (12) in Proposition 2.8 below.

Remark 1.7. Let us exhibit a couple of examples of Ω, Γ_D and Γ_N for which the condition (12) is satisfied.

(i) If $\overline{\Gamma_{\rm D}} \cap \overline{\Gamma_{\rm N}} = \emptyset$ and Γ is of class $C^{1,1}$, then (12) is satisfied (see Theorem 2.2.2.3 and Theorem 2.2.2.5 of [33]).
(ii) Let Ω be convex. If $\Gamma_{\rm D} = \Gamma$ or $\Gamma_{\rm D} = \emptyset$, then (12) is satisfied (see Theorem 3.2.1.2 and Theorem 3.2.1.3 of [33]).
(iii) If $n = 1$ or if Ω is a rectangle in $\mathbb{R}^2$ and $\Gamma_{\rm D}$ is a union of some of four edges of Γ, then (12) is satisfied. Indeed, since the weak solution $u = A_1^{-1} g$ can be extended to an open neighborhood of $\overline{\Omega}$ by reflection, it follows that $u = A_1^{-1} g \in H^2(\Omega)$.

Our existence result reads,

Theorem 1.8 *(Existence). We suppose that the condition* (12) *holds true. Let $T > 0$, $u_0 \in V$ and $f \in L^2(Q_T)$ be given and suppose that*

$$A_0 u_0 \in \mathcal{M}(\overline{\Omega}) \quad \text{and} \quad (A_0 u_0)_+ \in L^2(\Omega), \tag{13}$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of signed Radon measures (see also Notation in §1.3). In addition, assume that there exists $f^* \in L^2(\Omega)$ satisfying

$$f(x,t) \le f^*(x) \quad \text{a.e. in } Q_T. \tag{14}$$

Then there exists a strong solution $u = u(x,t)$ to the problem (6)–(9) *on* $[0, T]$.

Remark 1.9 *(Assumption (13)).*

(i) To be precise, the first half of (13) means that there exists $\mu \in \mathcal{M}(\overline{\Omega})$ such that

$$\langle A_0 u_0, w \rangle_V = \int_{\overline{\Omega}} w \, \mathrm{d}\mu \quad \text{for all } w \in V \cap C(\overline{\Omega}).$$

The second half of (13) means that the positive part μ_+ of μ is absolutely continuous (with respect to Lebesgue measure) with an $L^2(\Omega)$ density function. On the other hand, the negative part μ_- of μ may have a singular part. According to Theorem 1.8, such a singularity of the negative part disappears by the evolution of solutions. In such a point of view, a smoothing effect *partially* occurs.
(ii) If $u_0 \in V \cap H^2(\Omega)$ and $(\partial_\nu u_0)_+ = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_N, then (13) holds. Indeed, $A_0 u_0$ can be identified with $\mu \in \mathcal{M}(\overline{\Omega})$ defined by

$$\int_{\overline{\Omega}} w \, \mathrm{d}\mu = -\int_{\Omega} w \Delta u_0 \, \mathrm{d}x + \int_{\Gamma_N} (\partial_\nu u_0) w \, \mathrm{d}\mathcal{H}^{n-1} \quad \text{for all } w \in C(\overline{\Omega}).$$

Hence μ_+ is characterized by

$$\int_{\overline{\Omega}} w \, \mathrm{d}\mu_+ = \int_{\Omega} w (\Delta u_0)_- \, \mathrm{d}x + \int_{\Gamma_N} (\partial_\nu u_0)_+ w \, \mathrm{d}\mathcal{H}^{n-1}$$

for all $w \in C(\overline{\Omega})$ satisfying $w \geq 0$. Thus for $u_0 \in V \cap H^2(\Omega)$, (13) is equivalent to $(\partial_\nu u_0)_+|_{\Gamma_N} = 0$ (namely, the singular part of μ_+ vanishes). Moreover, the density function of μ_+ is $(\Delta u_0)_-$.

Remark 1.10 *(Assumptions on f).* Condition (14) is weaker than $f \in L^\infty(Q_T)$ or $f \in W^{1,1}(0,T;L^2(\Omega))$ (cf. [8]). Indeed, if $f \in W^{1,1}(0,T;L^2(\Omega))$, then $f^*(x) := f(x,0) + \int_0^T |\partial_t f(x,t)|\,\mathrm{d}t$ belongs to $L^2(\Omega)$ and satisfies (14). On the other hand, (14) is stronger than $f_+ := f \vee 0 \in L^\infty(0,T;L^2(\Omega))$. In fact, (14) yields $f_+ \in L^\infty(0,T;L^2(\Omega))$. However, even if $f_+ \in L^\infty(0,T;L^2(\Omega))$, (14) might not hold true. One may easily find a counterexample, e.g., $f(x,t) = |x-t|^{-\alpha}$, $\Omega = (0,1)$, $T = 1$ and $0 < \alpha < 1/2$.

The following theorem is concerned with a *comparison principle* for strong solutions of (6)–(9):

Theorem 1.11 *(Comparison principle). Let $T > 0$ and suppose that (12) is satisfied. For each $i = 1,2$, let $u_0^i \in X \cap V$ and $f^i \in L^2(Q_T)$ be such that there exists $f^* \in L^2(\Omega)$ satisfying*

$$f^1(x,t) \vee f^2(x,t) \leq f^*(x) \quad \text{a.e. in } Q_T.$$

For $i = 1,2$, let $u^i = u^i(x,t)$ be the unique strong solution of (6)–(9) with $u_0 = u_0^i$ and $f = f^i$ on $[0,T]$. If $u_0^1 \leq u_0^2$ a.e. in Ω and $f^1 \leq f^2$ a.e. in Q_T, then $u^1 \leq u^2$ a.e. in Q_T.

The comparison theorem stated above will be used to identify the limit of each solution $u = u(x,t)$ as $t \to \infty$.

Theorem 1.12 *(Convergence of solutions as $t \to \infty$). Let $u_0 \in X \cap V$ and assume that (12) holds and that*

(H1) $\mathcal{H}^{n-1}(\Gamma_D) > 0$;
(H2) *there exists a function $f_\infty \in L^2(\Omega)$ such that $f - f_\infty$ belongs to $L^2(0,\infty;L^2(\Omega))$;*
(H3) *$f \in L^\infty(0,\infty;L^2(\Omega))$, and (14) is satisfied.*

Then the unique solution $u = u(x,t)$ of (6)–(9) on $[0,\infty)$ converges to a function $z = z(x) \in X \cap V$ strongly in V as $t \to \infty$. Moreover, the limit z satisfies

$$z \geq u_0 \quad \text{and} \quad -\Delta z \geq f_\infty \ \text{ a.e. in } \Omega.$$

In addition, if $f(x,t) \leq f_\infty(x)$ for a.e. $(x,t) \in Q$, then the limit z coincides with the unique solution $\bar{z} \in X \cap V$ of the following variational inequality:

$$(\mathrm{VI})(u_0, f_\infty) \begin{cases} \bar{z} \in K_0(u_0) := \{v \in V : v \geq u_0 \ \text{ a.e. in } \Omega\}, \\ \displaystyle\int_\Omega \nabla \bar{z} \cdot \nabla (v - \bar{z})\,\mathrm{d}x \geq \int_\Omega f_\infty (v - \bar{z})\,\mathrm{d}x \quad \text{for all } v \in K_0(u_0). \end{cases}$$

Remark 1.13. Assumption (H1) is essentially required to ensure the convergence of the solution $u = u(x,t)$ as $t \to \infty$. Indeed, suppose that $\Gamma_{\mathrm{D}} = \emptyset$ (i.e., $\Gamma_{\mathrm{N}} = \Gamma$) and set $u_0(x) \equiv 1$ and $f(x,t) \equiv 1$. The unique strong solution of (6)–(9) is given by

$$u(x,t) = 1 + t \quad \text{for } (x,t) \in Q,$$

and then, $u(x,t)$ is divergent to ∞ at each $x \in \Omega$ as $t \to \infty$.

1.3. Outline of the paper and notation

In this subsection, we shall explain an idea of proving existence of solution and also give an outline of the present paper. In order to prove Theorem 1.8, we shall exploit a backward-Euler scheme for (4),

$$\frac{u_k - u_{k-1}}{\tau_k} + \alpha\left(\frac{u_k - u_{k-1}}{\tau_k}\right) - \Delta u_k \ni f_k \text{ in } L^2(\Omega), \quad u_k \in X \cap V \tag{15}$$

for $k = 1, \ldots, m$ through a minimization problem of the functional

$$J_k(v) := \frac{1}{2\tau_k}\int_\Omega |v|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \left\langle \frac{u_{k-1}}{\tau_k} + f_k, v \right\rangle_V \quad \text{for } v \in V$$

subject to a constraint

$$v \in K_0^k := \{v \in V : v \geq u_{k-1} \text{ a.e. in } \Omega\}.$$

Here we stress that the discretized equation (15) is posed in $L^2(\Omega)$, and it is a stronger form of the Euler–Lagrange equation for J_k, which is posed in V^*. Indeed, we shall prove existence of an L^2 solution (or strong solution) for (4), which also solves (1) by *pointwise* equivalence between two equations. Therefore we shall construct solutions to (15) in the strong form rather than the weak one. However, to this end, we need to verify further regularity of u_k for (15), e.g., $u_k \in H^2(\Omega)$, which is a classical fact for linear uniform elliptic problems but not obvious for (15) with such severe nonlinearity. Moreover, (4) (or (15)) involves two unbounded operators in $L^2(\Omega)$, that is, α and the Laplacian. In order to construct L^2 solutions, we need to control one of them. Indeed, it is possible if one carries out an energy technique by differentiating equation in time. However, we shall here avoid this strategy in order to relax the assumption on f. To overcome such difficulties, we shall rewrite the equation (15) as an elliptic variational inequality of obstacle type,

$$\frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k \leq f_k, \quad u_k \geq u_{k-1} \text{ in } \Omega,$$
$$\left(\frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k\right)(u_k - u_{k-1}) = 0 \text{ in } \Omega,$$

through the minimization problem and develop a regularity theory for solutions of the obstacle problem. It is already done in L^p spaces for $p > n/2$ (see, e.g., [37], [34]), however, to the authors' knowledge, there is no literature with an explicit description for $p \le n/2$, which is essentially required in our analysis. Furthermore, we shall also derive the so-called *Lewy–Stampacchia inequality*, which corresponds to elliptic estimates for linear uniform elliptic equations, and it enables us to control unbounded operators. The discretization scheme proposed here along with the regularity theory also play an important role to prove a *comparison principle*, which is more difficult to prove directly for (1) and (4), and to investigate long-time behaviors of solutions, in particular, revealing convergence of solutions and limiting equation.

The organization of the rest of the paper is as follows. Section 2 is devoted to developing the regularity theory as well as a comparison theorem for variational inequalities of obstacle type. In Section 3, we discuss a rigorous reduction of (1) to the evolution equation (4) of doubly nonlinear type in $L^2(\Omega)$ and prove the uniqueness of solutions for the initial-boundary value problem. In Section 4, we carry out the backward-Euler time-discretization (15) of (4) and construct a strong solution of (1) by establishing a new a priori estimate based on the regularity theory developed in Section 2. A comparison theorem for (1) is also proved. The long-time behavior of solutions will be investigated in Section 5. In the last section, we shall discuss other equivalent formulations of solutions for (1).

Notation. For each normed space N, we denote by N' the dual space of N with duality pairing $\langle g, v\rangle_N := {}_{N'}\langle g, v\rangle_N = g(v)$ for $v \in N$ and $g \in N'$. For Banach spaces U and W, the set of all bounded linear operators from U to W is denoted by $B(U, W)$. Moreover, the set of all linear topological isomorphisms from U to W is denoted by $\mathrm{Isom}(U, W)$, that is, $A \in \mathrm{Isom}(U, V)$ means that A is bijective from U to W, $A \in B(U, W)$ and $A^{-1} \in B(W, U)$. Furthermore, $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure in $\mathbb{R}^n$ for $k = 1, 2, \dots, n$. We also write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$. Moreover, $(a)_+ := a \vee 0$ and $(a)_- := (-a) \vee 0$ for $a \in \mathbb{R}$. Hereafter, C denotes a non-negative constant independent of the elements of the corresponding space and set and may vary from line to line.

For any bounded domain $\Omega \subset \mathbb{R}^n$, $\mathcal{M}(\overline{\Omega})$ denotes the set of signed Radon measures (i.e., finite Borel measures) on $\overline{\Omega}$. We recall that $\mathcal{M}(\overline{\Omega})$ is identified with the dual space $C(\overline{\Omega})'$ of $C(\overline{\Omega}) := \{f : \overline{\Omega} \to \mathbb{R}\colon f \text{ is continuous}\}$ with the norm $\|f\|_{C(\overline{\Omega})} := \sup_{x\in\overline{\Omega}} |f(x)|$. More precisely, by Riesz's representation theorem, for every $\xi \in C(\overline{\Omega})'$ there exists a unique $\mu \in \mathcal{M}(\overline{\Omega})$ such that

$$\langle \xi, f\rangle_{C(\overline{\Omega})} = \int_{\overline{\Omega}} f \,\mathrm{d}\mu \quad \text{for } f \in C(\overline{\Omega}).$$

By the Hahn–Jordan decomposition, every $\mu \in \mathcal{M}(\overline{\Omega})$ is uniquely decomposed into two positive Radon measures denoted by $\mu_\pm \in \mathcal{M}(\overline{\Omega})$ such that μ_+ and μ_- are *mutually singular* (or *singular*, *orthogonal*, that is, there exist two Borel sets $A, B \subset \overline{\Omega}$ such that $A \cap B = \emptyset$, $A \cup B = \overline{\Omega}$ and $\mu_+(A) = \mu_-(B) = 0$) and $\mu = \mu_+ - \mu_-$. Furthermore, for $1 \le p \le \infty$, any $g \in L^p(\Omega)$ can be identified with the absolutely continuous (with respect to Lebesgue measure) $\mu_g \in \mathcal{M}(\overline{\Omega})$ given by

$$\mu_g(B) = \int_B g \,\mathrm{d}x \quad \text{for Borel sets } B \subset \overline{\Omega}$$

(hence g is a density function of μ_g), which can be also identified with $\xi_g \in C(\overline{\Omega})'$ in the following sense:

$$\langle \xi_g, f \rangle_{C(\overline{\Omega})} = \int_{\overline{\Omega}} f \,\mathrm{d}\mu_g = \int_{\Omega} fg \,\mathrm{d}x \quad \text{for } f \in C(\overline{\Omega}).$$

For $\mu \in \mathcal{M}(\overline{\Omega})$ and $g \in L^p(\Omega)$, we write $\mu + g$ instead of $\mu + \mu_g \in \mathcal{M}(\overline{\Omega})$, if no confusion may arise. Hence $L^p(\Omega) \subset \mathcal{M}(\overline{\Omega}) \simeq C(\overline{\Omega})'$. Conversely, for $\mu \in \mathcal{M}(\overline{\Omega})$, we write $\mu \in L^p(\Omega)$, provided that μ is absolutely continuous (with respect to Lebesgue measure) with a density function $g \in L^p(\Omega)$, i.e., $\mu(B) = \int_B g \,\mathrm{d}x$ for Borel sets $B \subset \overline{\Omega}$.

2. Regularity theory for variational inequalities of obstacle type

In this section, based on the approach of Gustafsson [34], we revisit a regularity theory for variational inequalities of obstacle type. In classical literature on variational inequalities of obstacle type, the $W^{2,p}(\Omega)$-regularity of solutions is often obtained by using a penalization technique (see, e.g., [37, Chap. 4], [27]). On the other hand, Gustafsson [34] gave a simpler alternative proof by introducing an auxiliary variational inequality and by proving the coincidence of solutions for both problems. Let us also remark that, in previous results, it is assumed that $W^{2,p}(\Omega) \subset C(\overline{\Omega})$ (namely, $p > n/2$) in order to utilize the classical maximum principle for linear elliptic equations.

We shall establish a $W^{2,p}(\Omega)$-regularity result for variational inequalities of obstacle type equipped with a mixed boundary condition by properly modifying the argument of Gustafsson [34]. It is noteworthy that we do not assume that $W^{2,p}(\Omega) \subset C(\overline{\Omega})$, i.e., $p > n/2$, as we employ Stampacchia's truncation technique instead of the classical maximum principle. Indeed, we essentially need to apply the theory for $p = 2$, and moreover, it is also meaningful as an independent interest to explicitly describe a regularity theory in any L^p framework.

2.1. Main results of this section

For $\sigma \geq 0$, let $A := A_\sigma \in B(V, V')$ be defined as in (11) (or (16) below). Define a symmetric bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ associated with A by

$$a(u, v) := \langle Au, v \rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \sigma uv) \,\mathrm{d}x \quad \text{for } u, v \in V. \tag{16}$$

Throughout this section, we assume that

$$\sigma > 0 \quad \text{if} \quad \mathcal{H}^{n-1}(\Gamma_\mathrm{D}) = 0, \tag{17}$$

which also means that $\mathcal{H}^{n-1}(\Gamma_\mathrm{D}) > 0$ if $\sigma = 0$. Under the condition (17), (by the Poincaré inequality for the case that $\sigma = 0$), $a(\cdot,\cdot)$ turns out to be coercive on $V \times V$. Hence A is invertible, and A^{-1} belongs to $B(V', V)$. Let $f \in V'$ and $\psi \in V$ and define a closed convex subset K_0 of V by

$$K_0 := \{v \in V : v \geq \psi \text{ a.e. in } \Omega\}. \tag{18}$$

We also define a functional J on V by

$$J(v) := \frac{1}{2}a(v,v) - \langle f, v\rangle_V \quad \text{for } v \in V. \tag{19}$$

In this section, we shall discuss regularity of minimizers for the problem,

$$\text{Minimize } J(v) \text{ subject to } v \in K_0. \tag{20}$$

As we shall show in §2.2, problem (20) is equivalent to the following variational inequality,

$$u \in K_0, \quad a(u, v-u) \geq \langle f, v-u\rangle_V \text{ for all } v \in K_0 \tag{21}$$

(see Theorems 2.3 and 2.4, which also provide other formulations equivalent to (20) and (21)). To this end, we sequentially introduce a couple of assumptions with some related remarks below.

(I): Let $p \in \mathbb{R}$ satisfy

$$1 < p < \infty, \quad p \geq \frac{2n}{n+2}. \tag{22}$$

Since the Hölder conjugate $q := p/(p-1)$ of p satisfies $q \leq 2^* := 2n/(n-2)$ if $n \geq 3$, and Ω is a Lipschitz domain, by Sobolev's embedding theorem, the continuous embeddings $V \hookrightarrow L^q(\Omega)$ and $L^p(\Omega) \cong (L^q(\Omega))' \hookrightarrow V'$ hold true. We also note that $W^{2,p}(\Omega)$ is continuously embedded in $H^1(\Omega)$ by (22).

(II): As for f and ψ, we suppose that

$$f \in L^p(\Omega), \quad \psi \in V. \tag{23}$$

In addition, let us also assume that

$$A\psi \in \mathcal{M}(\overline{\Omega}) \text{ and } (A\psi)_+ \in L^p(\Omega). \tag{24}$$

Then $A\psi - f$ is identified with some $\mu \in \mathcal{M}(\overline{\Omega})$ such that

$$\langle A\psi - f, \varphi\rangle_V = \int_{\overline{\Omega}} \varphi \, d\mu = \int_{\overline{\Omega}} \varphi \, d\mu_+ - \int_{\overline{\Omega}} \varphi \, d\mu_- = \int_{\Omega} \varphi (A\psi - f)_+ \, dx - \int_{\overline{\Omega}} \varphi \, d\mu_- \tag{25}$$

for all $\varphi \in V \cap C(\overline{\Omega})$. Here, by Lemma A.1 in Appendix, μ_+ is absolutely continuous (with respect to Lebesgue measure) and has an $L^p(\Omega)$ density function (simply denoted by $(A\psi - f)_+$). See also Remark A.3 for concrete examples of ψ satisfying (24).

(III): Let us also introduce the following assumption:

$$A_1^{-1} g \in W^{2,p}(\Omega) \quad \text{for all } g \in L^p(\Omega), \tag{26}$$

(here we note that (12) is a special case of (26) with $p = 2$). Condition (26) can be regarded as an *elliptic regularity* of weak solutions for the elliptic boundary value problem,

$$-\Delta u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u = 0 \text{ on } \Gamma_N,$$

and it holds true in many cases, e.g., smooth domains with $\Gamma_N = \emptyset$ or $\Gamma_D = \emptyset$ (see, e.g., [32]). However, the validity of (26) is more delicate, if Ω is not smooth or mixed boundary conditions are imposed. So we here explicitly assume it.

To take account of boundary conditions, we define a subspace of $W^{2,p}(\Omega)$ by

$$X^p := \{v \in W^{2,p}(\Omega) : \ \gamma_0(\nabla v) \cdot \nu = 0 \ \ \mathcal{H}^{n-1}\text{-a.e. on } \Gamma_N\}.$$

We are now in position to state main result of this section. The following theorem will be used for proving Theorem 1.8 in Section 4.

Theorem 2.1 *(Regularity of solutions for variational inequalities of obstacle type). Assume that* (17)*,* (22)*,* (23)*,* (24) *and* (26) *are satisfied. Problem* (20) (*equivalently,* (21)) *admits a unique solution* $u \in V$*. Moreover, it holds that*

$$u \in X^p \cap K_0, \quad f \le Au \le f \vee \hat{f} \ \text{ a.e. in } \Omega. \tag{27}$$

Here the inequality above is the so-called Lewy–Stampacchia inequality (see [42]).

We next give a comparison theorem for variational inequalities of obstacle type.

Theorem 2.2 *(Comparison principle for variational inequalities of obstacle type). We suppose that* (17) *and* (22) *are satisfied. For* $i = 1, 2$*, let* (f_i, ψ_i) *satisfy* (23) *and* (24) *and set* $K_0^i := \{v \in V : v \ge \psi_i \text{ a.e. in } \Omega\}$*. Let* $u_i \in V$ *be the unique solution of the variational inequality*:

$$u_i \in K_0^i, \quad a(u_i, v - u_i) \ge \langle f_i, v - u_i\rangle_V \ \text{ for all } \ v \in K_0^i \tag{28}$$

for $i = 1, 2$*. If* $f_1 \le f_2$ *and* $\psi_1 \le \psi_2$ *a.e. in* Ω*, then* $u_1 \le u_2$ *a.e. in* Ω*.*

This theorem will be used to prove Theorem 1.11, a comparison theorem for the evolutionary problem (6), with the aid of a discretization argument.

2.2. *Proof of Theorem 2.1*

Let us further set up notation. Define

$$K_1 := \{v \in V : Av \ge f \text{ in } V'\}.$$

Here the inequality $Av \ge f$ in V' means that $\langle Av - f, \varphi\rangle_V \ge 0$ for all $\varphi \in V$ satisfying $\varphi \ge 0$ a.e. in Ω. Moreover, define $\hat{J} : V \to \mathbb{R}$ by

$$\hat{J}(v) := \frac{1}{2} a(v, v) - \langle \hat{f}, v\rangle_V \quad \text{ for } \ v \in V,$$

where $\hat{f} := A\psi \in V'$. In what follows, we shall give several equivalent forms to (20) and (21), and they will provide additional information of solutions to (20) and (21).

Proposition 2.3. *Suppose* (17) *and let $f \in V'$ and $\psi \in V$. Then the following five conditions for $u \in V$ are equivalent to each other*:

(a) $u \in K_0$, $J(u) \le J(v)$ *for all* $v \in K_0$,
(b) $u \in K_0$, $a(u, v-u) \ge \langle f, v-u \rangle_V$ *for all* $v \in K_0$,
(c) $u \in K_0 \cap K_1$, $\langle Au - f, u - \psi \rangle_V = 0$,
(d) $u \in K_1$, $a(u, v-u) \ge \langle \hat{f}, v-u \rangle_V$ *for all* $v \in K_1$,
(e) $u \in K_1$, $\hat{J}(u) \le \hat{J}(v)$ *for all* $v \in K_1$.

Moreover, there exists a unique element $u \in V$ satisfying all the conditions.

Proof. Since J is a coercive, continuous, strictly convex functional on the closed convex set K_0, J admits a unique minimizer u over K_0. Hence u satisfies (a).

We shall prove the equivalence of the conditions (a)–(e). It is well known (see, e.g., [37]) that (a) $\Leftrightarrow$ (b) and (d) $\Leftrightarrow$ (e). So, let us here start with showing that (b) $\Rightarrow$ (c). The condition (b) is equivalently rewritten by

$$u \in K_0, \quad \langle Au - f, v - u \rangle_V \ge 0 \quad \text{for all } v \in K_0. \tag{29}$$

For any $\varphi \in V$ with $\varphi \ge 0$ a.e. in Ω, substituting $v = u + \varphi \in K_0$ to (29), we have $\langle Au - f, \varphi \rangle_V \ge 0$, which yields that $u \in K_1$. On the other hand, substitute $v = \psi \in K_0$ and $v = 2u - \psi \in K_0$ to (29). Then one can obtain $\langle Au - f, u - \psi \rangle_V = 0$. Hence (c) holds.

To prove the inverse relation, (c) $\Rightarrow$ (b), let u satisfy (c). For any $v \in K_0$, we see that

$$a(u, v-u) - \langle f, v-u \rangle_V = \langle Au - f, v - \psi \rangle_V - \langle Au - f, u - \psi \rangle_V.$$

Since the first term of the right-hand side is nonnegative (by $u \in K_1$ and $v \in K_0$) and the second term vanishes (by the equation of (c)), the condition (b) follows.

We shall prove the equivalence between (c) and (d) in a similar fashion to the above. Firstly, suppose that u satisfies (c). For any $v \in K_1$, one finds that

$$\begin{aligned} a(u, v-u) - \langle \hat{f}, v-u \rangle_V &= \langle Av - Au, u \rangle_V - \langle Av - Au, \psi \rangle_V \\ &= \langle Av - f, u - \psi \rangle_V - \langle Au - f, u - \psi \rangle_V. \end{aligned}$$

Here we used the fact that $\langle Aw, z \rangle_V = \langle Az, w \rangle_V$ for all $w, z \in V$. Noting that the right-hand side is non-negative by (c) and the fact that $u \in K_0$ and $v \in K_1$, one can get (d). To check the inverse relation, we also rewrite (d) as

$$u \in K_1, \quad \langle Av - Au, u - \psi \rangle_V \ge 0 \quad \text{for all } v \in K_1. \tag{30}$$

For any $\varphi \in L^2(\Omega)$ with $\varphi \ge 0$ a.e. in Ω, substituting $v = u + A^{-1}\varphi \in K_1$ to (30), we have $(\varphi, u - \psi)_{L^2(\Omega)} \ge 0$, which along with the arbitrariness of $\varphi \ge 0$ implies $u \in K_0$. Moreover, let us also substitute $v = A^{-1} f \in K_1$ and $v = 2u - A^{-1} f \in K_1$ in (30). Then we obtain $\langle Au - f, u - \psi \rangle_V = 0$, whence (c) follows. Consequently, all the conditions (a)–(e) are equivalent. □

Moreover, under (23) and (24), introduce a closed (in V) convex set K_2 given by

$$K_2 := \{v \in V : f \le Av \le f \vee \hat{f} \text{ in } V'\} \subset K_1.$$

Here note that $(\hat{f} - f)_+ \in L^p(\Omega)$ (by (23), (24) and Lemma A.2) and that $f \vee \hat{f} := (\hat{f} - f)_+ + f \in L^p(\Omega)$. Then by the Hahn–Banach theorem and Riesz's representation theorem, there uniquely exists $g_v \in L^p(\Omega)$ such that

$$\langle Av, w\rangle_V = \int_\Omega g_v w \, \mathrm{d}x \qquad \text{for all } \ w \in V \cap L^q(\Omega).$$

Hence we simply write

$$Av \in L^p(\Omega) \qquad \text{for all } \ v \in K_2. \tag{31}$$

The following theorem is a key of proving Theorem 2.1.

Theorem 2.4. *Suppose that* (17), (22), (23) *and* (24) *are satisfied. Then each of the following conditions* (f)–(h) *is equivalent to the conditions* (a)–(e) *of Proposition* 2.3:

(f) $u \in K_2$, $\hat{J}(u) \le \hat{J}(v)$ *for all* $v \in K_2$,
(g) $u \in K_2$, $a(u, v-u) \ge \langle \hat{f}, v-u\rangle_V$ *for all* $v \in K_2$,
(h) $u \in K_0 \cap K_2$, $(Au - f)(u - \psi) = 0$ *a.e. in* Ω.

To prove Theorem 2.4, we first prove a couple of lemmas, which will be also used in later sections.

Lemma 2.5. *Let* $w \in H^1(\Omega)$ *and set* $w_+(x) := (w(x))_+$ *for* $x \in \Omega$. *Then* $w_+ \in H^1(\Omega)$ *and* $\gamma_0(w_+) = (\gamma_0 w)_+$ $\mathcal{H}^{n-1}$*-a.e. on* Γ.

Proof. Set $\Omega_+ := \{x \in \Omega : w(x) > 0\}$ and recall Theorem A.1. of [37] to observe that $w_+ \in H^1(\Omega)$ and

$$w_+ = \begin{cases} w & \text{a.e. in } \Omega_+, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_+, \end{cases} \qquad \nabla w_+ = \begin{cases} \nabla w & \text{a.e. in } \Omega_+, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_+. \end{cases} \tag{32}$$

Set $W := C(\overline{\Omega}) \cap H^1(\Omega)$. Since W is dense in $H^1(\Omega)$, there exists a sequence $\{w_n\}$ in W such that $w_n \to w$ strongly in $H^1(\Omega)$ as $n \to \infty$. Noting that

$$\|(w_n)_+ - w_+\|_{L^2(\Omega)} \le \|w_n - w\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad n \to \infty, \tag{33}$$

we observe that $(w_n)_+ \to w_+$ strongly in $L^2(\Omega)$. Applying (32) to w_n, we also have

$$\|(w_n)_+\|^2_{H^1(\Omega)} = \|(w_n)_+\|^2_{L^2(\Omega)} + \|\nabla (w_n)_+\|^2_{L^2(\Omega)} \le \|w_n\|^2_{L^2(\Omega)} + \|\nabla w_n\|^2_{L^2(\Omega)} = \|w_n\|^2_{H^1(\Omega)}.$$

Since $\{w_n\}$ is bounded in $H^1(\Omega)$, so is $\{(w_n)_+\}$. Hence, one can extract a (non-relabeled) subsequence of $\{n\}$ such that $(w_n)_+ \to w_+$ weakly in $H^1(\Omega)$ Again from (32), we have

$$\|(w_n)_+\|^2_{H^1(\Omega)} = \|(w_n)_+\|^2_{L^2(\Omega)} + (\nabla(w_n)_+,\ \nabla w_n)_{L^2(\Omega)}$$
$$\to \|w_+\|^2_{L^2(\Omega)} + (\nabla w_+,\ \nabla w)_{L^2(\Omega)} = \|w_+\|^2_{H^1(\Omega)},$$

which together with the uniform convexity of $H^1(\Omega)$ also implies that $(w_n)_+ \to w_+$ strongly in $H^1(\Omega)$. Since $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$, we particularly deduce that $\gamma_0 w_n \to \gamma_0 w$ and $\gamma_0(w_n)_+ \to \gamma_0 w_+$ strongly in $L^2(\Gamma)$. As in (33), one can verify that $(\gamma_0 w_n)_+ \to (\gamma_0 w)_+$ strongly in $L^2(\Gamma)$. On the other hand, since $w_n \in C(\overline{\Omega})$, it is clear that

$$\gamma_0(w_n)_+ = (\gamma_0 w_n)_+ \quad \mathcal{H}^{n-1}\text{-a.e. on } \Gamma. \tag{34}$$

Passing to the limit as $n \to \infty$ in (34), we conclude that $\gamma_0 w_+ = (\gamma_0 w)_+$ $\mathcal{H}^{n-1}$-a.e. on Γ. □

Lemma 2.6. *If $v_1, v_2 \in V$, then $v_1 \vee v_2 \in V$ and $v_1 \wedge v_2 \in V$.*

Proof. Applying Lemma 2.5 to $w = \pm(v_1 - v_2)$, we find that $(v_1 - v_2)_+$ and $(v_2 - v_1)_+$ belong to V. Hence we obtain $v_1 \vee v_2 \in V$ and $v_1 \wedge v_2 \in V$, since $v_1 \vee v_2 = v_2 + (v_1 - v_2)_+$ and $v_1 \wedge v_2 = v_2 - (v_2 - v_1)_+$. □

Let us move on to a proof of Theorem 2.4.

Proof of Theorem 2.4. It is obvious that (f) ⇔ (g). As in Proposition 2.3, one can uniquely choose $u \in K_2$ which satisfies (f) and (g). Next, we shall prove the equivalence between (a)–(e) and (f), (g). Let u_1 be the unique element of V satisfying (a)–(e) and let u_2 be the unique element of V satisfying (f) and (g). We claim that $u_1 = u_2$. Indeed, note that $Au_2 \in L^p(\Omega)$ by $u_2 \in K_2$ (see (31)). Set $w := u_2 - \psi \in V$. Since u_2 satisfies (g), it follows that

$$0 \le a(u_2, v - u_2) - \langle \hat{f}, v - u_2\rangle_V = \langle Au_2 - A\psi, v - u_2\rangle_V$$
$$= \langle Av - Au_2, w\rangle_V \quad \text{for all } v \in K_2. \tag{35}$$

We set a measurable set

$$N := \{x \in \Omega :\ w(x) < 0\}.$$

Moreover, define $g \in L^p(\Omega)$ (see (31)) by

$$g(x) := \begin{cases} \left(f \vee \hat{f}\right)(x) & \text{if } x \in N, \\ Au_2(x) & \text{if } x \in \Omega \setminus N. \end{cases}$$

Then by definition one can observe that $A^{-1}g \in K_2$. Hence substituting $v = A^{-1}g$ to (35), we have

$$0 \le \langle g - Au_2, w \rangle_V = \int_N \Big((f \vee \hat{f}) - Au_2 \Big) w \, \mathrm{d}x.$$

Since $(f \vee \hat{f}) - Au_2 \ge 0$ and $w < 0$ a.e. in N, one can derive the relation $Au_2 = f \vee \hat{f}$ a.e. in N. Hence

$$\langle Au_2 - f \vee \hat{f}, w_- \rangle_V = \int_\Omega (Au_2 - f \vee \hat{f}) w_- \, \mathrm{d}x = \int_N (Au_2 - f \vee \hat{f}) w_- \, \mathrm{d}x = 0, \tag{36}$$

which will be used later.

Recalling that

$$w_- \in V, \quad w_- = \begin{cases} -w & \text{in } N, \\ 0 & \text{in } \Omega \setminus N, \end{cases} \quad \nabla w_- = \begin{cases} -\nabla w & \text{a.e. in } N, \\ 0 & \text{a.e. in } \Omega \setminus N \end{cases}$$

(by Lemma 2.6 and Theorem A.1. of [37]), we observe that

$$\begin{aligned} 0 &\le a(w_-, w_-) = -a(w, w_-) = \langle Aw, -w_- \rangle_V = \langle Au_2 - \hat{f}, -w_- \rangle_V \\ &= \langle Au_2 - f \vee \hat{f}, -w_- \rangle_V + \langle f \vee \hat{f} - \hat{f}, -w_- \rangle_V \overset{(36)}{=} \langle f \vee \hat{f} - \hat{f}, -w_- \rangle_V =: I. \end{aligned} \tag{37}$$

We claim that $I \le 0$. Indeed, taking a smooth approximation $w_-^\varepsilon \in V \cap C(\overline{\Omega})$ such that $w_-^\varepsilon \ge 0$ and $w_-^\varepsilon \to w_-$ weakly in V as $\varepsilon \to 0$ (by density assumption), we find that

$$\langle f \vee \hat{f} - \hat{f}, -w_-^\varepsilon \rangle_V = \langle (\hat{f} - f)_+ + f - \hat{f}, -w_-^\varepsilon \rangle_V \overset{(25)}{=} - \int_{\overline{\Omega}} w_-^\varepsilon \, \mathrm{d}\mu_- \le 0.$$

Thus letting $\varepsilon \to 0$, we obtain $I \le 0$, and hence, we deduce by (37) that $a(w_-, w_-) = 0$, which along with the coercivity of $a(\cdot, \cdot)$ implies that $w_- = 0$ (i.e., $w \ge 0$) a.e. in Ω. Therefore u_2 belongs to K_0.

Substitute $v = A^{-1} f \in K_2$ to the condition (35). Then we obtain

$$\langle Au_2 - f, u_2 - \psi \rangle_V \le 0. \tag{38}$$

Moreover, noting that $Au_2 - f \ge 0$ (by $u_2 \in K_2$) and $u_2 - \psi \ge 0$ (by $u_2 \in K_0$), we derive $\langle Au_2 - f, u_2 - \psi \rangle_V = 0$ by (38). Hence, $u = u_2$ satisfies the condition (c). By uniqueness, we obtain $u_1 = u_2$. Thus we have proved that all the conditions (a)–(g) are equivalent.

Finally, we note that (h) immediately implies (c), since $K_2 \subset K_1$. Conversely, let u satisfy (c). Then u belongs to K_2 by (f), and hence, $Au \in L^p(\Omega)$ (see (31)) and

$$0 = \langle Au - f, u - \psi \rangle_V = \int_\Omega (Au - f)(u - \psi) \, \mathrm{d}x.$$

Thus we obtain $(Au - f)(u - \psi) = 0$ a.e. in Ω, since $Au \ge f$ and $u \ge \psi$ a.e. in Ω. Therefore (h) holds. □

Remark 2.7. In the proof of Theorem 2.4, we used the density of $C(\overline{\Omega}) \cap V$ in V only for choosing an approximate sequence w_-^ε of w_-. If $\hat{f} = A\psi$ lies on $L^p(\Omega)$, one can derive the same conclusion (i.e., $I \le 0$) without taking an approximate sequence. Hence the density is not necessary for the case.

Thanks to Theorem 2.4, for each solution u of the variational inequality (21) of obstacle type, we have obtained an additional information, $u \in K_2$. In order to prove Theorem 2.1, we shall more explicitly clarify the feature of the additional information. To this end, let us start with giving equivalent conditions to the assumption (26).

Proposition 2.8. *Under the assumption (22), for $\lambda > 0$, the operator $A_\lambda|_{X^p \cap V}$ restricted onto $X^p \cap V$ is injective and bounded linear from $X^p \cap V$ into $L^p(\Omega)$, and it coincides with the operator $-\Delta + \lambda$, where Δ means the Laplace operator from $D(\Delta) = X^p \cap V$ into $L^p(\Omega)$, that is, the Laplacian equipped with the Dirichlet and Neumann boundary conditions on Γ_D and Γ_N, respectively, in a strong form.*

Moreover, the following conditions for Ω and Γ_D, Γ_N are equivalent to each other:

- (i) *there exists $\lambda > 0$ such that $A_\lambda^{-1} g \in W^{2,p}(\Omega)$ for all $g \in L^p(\Omega)$;*
- (ii) *for any $\lambda > 0$, it holds that $A_\lambda^{-1} g \in W^{2,p}(\Omega)$ for all $g \in L^p(\Omega)$;*
- (iii) *there exists $\lambda > 0$ such that $A_\lambda^{-1} g \in X^p$ for all $g \in L^p(\Omega)$, and $(-\Delta + \lambda) \in \mathrm{Isom}(X^p \cap V, L^p(\Omega))$;*
- (iv) *for any $\lambda > 0$, it holds that $A_\lambda^{-1} g \in X^p$ for all $g \in L^p(\Omega)$, and $(-\Delta + \lambda) \in \mathrm{Isom}(X^p \cap V, L^p(\Omega))$.*

Proof. Denote $B_\lambda := A_\lambda|_{X^p \cap V}$ for $\lambda > 0$. Then, for $u \in X^p \cap V$, we observe by Green's formula, which is valid for Lipschitz domains, that

$$\langle B_\lambda u, v\rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda u v)\,\mathrm{d}x = \int_\Omega (-\Delta u + \lambda u)\, v\,\mathrm{d}x \qquad \text{for all } v \in V \cap W^{1,q}(\Omega),$$

which implies that $B_\lambda u = -\Delta u + \lambda u$ and $B_\lambda \in B(X^p \cap V, L^p(\Omega))$, since $V \cap W^{1,q}(\Omega)$ is dense in $L^q(\Omega)$. Thus we obtain $B_\lambda = (-\Delta + \lambda)$. Moreover, B_λ is injective, since so is A_λ.

As for the equivalence of (i)–(iv), we shall show (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii). It is clear that (ii) $\Rightarrow$ (i) (and also (iv) $\Rightarrow$ (iii)). We show (i) $\Rightarrow$ (iii). Assume (i), let $g \in L^p(\Omega)$ and set $u := A_\lambda^{-1} g \in W^{2,p}(\Omega) \cap V$. For all $v \in V \cap W^{1,q}(\Omega)$, we have

$$\int_\Omega g v\,\mathrm{d}x = \langle A_\lambda u, v\rangle_V = \int_\Omega (\nabla u \cdot \nabla v + \lambda u v)\,\mathrm{d}x = \int_{\Gamma_N} (\partial_\nu u) v\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega (-\Delta u + \lambda u)\, v\,\mathrm{d}x,$$

which implies that $u \in X^p$ and $-\Delta u(x) + \lambda u(x) = g(x)$ for a.e. $x \in \Omega$. Hence $u = B_\lambda^{-1} g$. Therefore, B_λ is surjective from $X^p \cap V$ into $L^p(\Omega)$. By the open mapping theorem, we obtain $(-\Delta + \lambda) = B_\lambda \in \mathrm{Isom}(X^p \cap V, L^p(\Omega))$.

We next show (iii) $\Rightarrow$ (iv). Under the condition (iii), it holds that $B_\lambda \in \mathrm{Isom}(X^p \cap V, L^p(\Omega))$, in particular, $B_\lambda : X^p \cap V \to L^p(\Omega)$ is a Fredholm operator of index zero (see, e.g., [19]). For arbitrary $\mu > 0$, we find that $B_\mu = B_\lambda + (\mu - \lambda)$ is a Fredholm operator of index zero from

$X^p \cap V$ to $L^p(\Omega)$ as well, since $X^p \cap V$ is compactly embedded in $L^p(\Omega)$. Since B_μ is injective (i.e., $\dim\ker(B_\mu) = 0$), we infer that B_μ is surjective, and hence B_μ also belongs to $\mathrm{Isom}(X^p \cap V, L^p(\Omega))$. Furthermore, for any $g \in L^p(\Omega)$ and $\mu > 0$, the element $u = (-\Delta + \mu)^{-1} g = B_\mu^{-1} g$ belongs to $X^p \cap V$. Hence $B_\mu u = g$, i.e., $A_\mu u = g$, which implies $A_\mu^{-1} g = u \in X^p$. Thus (iv) follows.

It is obvious that (iv) implies (ii) by the definition of X^p. Thus we have shown that all the conditions (i)–(iv) are equivalent to each other. □

Then we can prove

Proposition 2.9. *Assume that* (17)*,* (22)*,* (23)*,* (24) *and* (26) *holds. Then*

$$K_2 \subset X^p \cap V \subset W^{2,p}(\Omega).$$

Proof. Let $v \in K_2$. Then $Av \in L^p(\Omega)$ (see (31)). In case $\sigma > 0$, the conclusion follows from Proposition 2.8 immediately. In case $\sigma = 0$ (then $\mathcal{H}^{n-1}(\Gamma_D) > 0$ by (17)), $B_0 = A_0|_{X^p \cap V}$ can be regarded as a sum of the Fredholm operator $B_1 : X^p \cap V \to L^p(\Omega)$ and a compact operator $-I : X^p \cap V \to L^p(\Omega)\,;\ u \mapsto -u$, and hence, B_0 has the same index as B_1. Since the index of B_1 is zero and B_0 is injective, namely, $\dim \mathrm{Ker}\, B_0 = 0$ (by (17)), we deduce that $B_0 : X^p \cap V \to L^p(\Omega)$ is surjective. Hence there exists $\bar{u} \in X^p \cap V$ such that $B_0 \bar{u} = Av \in L^p(\Omega)$. By (17) and $B_0 \bar{u} = A_0 \bar{u} = A\bar{u}$, we conclude that $v = \bar{u} \in X^p \cap V \subset W^{2,p}(\Omega)$. Thus $K_2 \subset X^p \cap V \subset W^{2,p}(\Omega)$. □

Remark 2.10. Under (26), the assumptions for ψ in (23) along with $A\psi \in L^p(\Omega)$ is equivalent to $\psi \in X^p \cap V$. Indeed, let $\psi \in V$ satisfy $A\psi \in L^p(\Omega)$. Then as in the proof of Proposition 2.9, one can check that $\psi \in W^{2,p}(\Omega)$. Moreover, by Green's formula, we find that

$$\begin{aligned}
\int_\Omega (A\psi) v \,\mathrm{d}x = \langle A\psi, v \rangle_V &= \int_\Omega \nabla\psi \cdot \nabla v \,\mathrm{d}x + \sigma \int_\Omega \psi v \,\mathrm{d}x \\
&= \int_\Omega (-\Delta\psi + \sigma\psi)\, v \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} (\partial_\nu \psi) v \,\mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

for all $v \in V \cap W^{1,q}(\Omega)$. Thus by the arbitrariness of v, we obtain $\partial_\nu \psi = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_N, whence follows $\psi \in X^p$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Existence and uniqueness of solutions are due to Proposition 2.3. By Theorem 2.4, the unique element $u \in V$ satisfying (a)–(h) belongs to K_2. Hence, by Proposition 2.9, one has $u \in X^p \cap V$. Thus the inequalities of (27) follow from the definition of K_2. □

2.3. *Proof of Theorem 2.2*

To prove Theorem 2.2, we prove the following lemma needed later.

Lemma 2.11. *We suppose that* (17)*,* (22)*,* (23) *and* (24) *are satisfied. Let $u \in V$ be the unique solution of* (a)–(h)*. Then it holds that $u \le w$ a.e. in Ω for all $w \in K_0 \cap K_1$ satisfying $Aw \in L^p(\Omega)$.*

Proof. We set $N := \{x \in \Omega : w(x) < u(x)\}$ and $v := u \wedge w$. Since $w \in K_0$, by Lemma 2.6 and (32), v satisfies

$$v \in K_0, \quad v = \begin{cases} u & \text{a.e. in } \Omega \setminus N, \\ w & \text{a.e. in } N, \end{cases} \qquad \nabla v = \begin{cases} \nabla u & \text{a.e. in } \Omega \setminus N, \\ \nabla w & \text{a.e. in } N. \end{cases}$$

Substituting v into the variational inequality (b) of Proposition 2.3, we have

$$0 \le \langle Au - f, v - u \rangle_V = \int_N (Au - f)(w - u)\, dx.$$

Since $Au - f \ge 0$ and $w - u < 0$ a.e. in N, it follows that $Au = f$ a.e. in N. Here we note that

$$v - u \in V, \quad v - u = \begin{cases} 0 & \text{a.e. in } \Omega \setminus N, \\ w - u < 0 & \text{a.e. in } N, \end{cases} \qquad \nabla(v - u) = \begin{cases} 0 & \text{a.e. in } \Omega \setminus N, \\ \nabla(w - u) & \text{a.e. in } N. \end{cases}$$

From the fact that $u - v = (u - w)_+$ and $Aw \in L^p(\Omega)$, we obtain

$$\begin{aligned} 0 \le a(u - v, u - v) &= a(u - w, u - v) = \langle A(u - w), u - v \rangle_V \\ &= \int_N (Au - Aw)(u - w)\, dx = \int_N (f - Aw)(u - w)\, dx. \end{aligned}$$

Since $Aw \ge f$ (by $w \in K_1$) and $u > w$ a.e. in N, we conclude that $a(u - v, u - v) = 0$, whence $u = v$ (hence $u \le w$) a.e. in Ω. □

The lemma above will also play a crucial role for identifying the limit of each solution $u = u(x, t)$ for (6)–(9) as $t \to \infty$ in the proof of Theorem 1.12. We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By assumption, we find that $K_0^2 \subset K_0^1$ and $K_1^2 \subset K_1^1$, where $K_1^j := \{v \in V : Av \ge f_j \text{ in } V'\}$ for $j = 1, 2$. Moreover, u_2 belongs to both K_0^1 and K_1^1, and $Au_2 \in L^p(\Omega)$ as well. By Lemma 2.11, we conclude that $u_1 \le u_2$ a.e. in Ω. □

3. Reduction to an evolution equation and the uniqueness of solution

In this section, we first reduce the problem (6)–(9) to the Cauchy problem for a nonlinear evolution equation in $L^2(\Omega)$ with the aid of convex analysis. Then we shall prove Theorem 1.6 on the uniqueness of solution.

Let us begin with reformulating (1) as a parabolic *inclusion* with a multivalued nonlinear operator acting on the time derivative of $u(x, t)$. Let $\alpha : \mathbb{R} \to 2^{\mathbb{R}}$ be given by

$$\alpha(s) = \begin{cases} \{0\} & \text{if } s > 0, \\ (-\infty, 0] & \text{if } s = 0 \end{cases} \tag{39}$$

with the domain $D(\alpha) = [0, \infty)$. Then $s + \alpha(s)$ is the (multi-valued) inverse mapping of the function $(s)_+$, and it can be also represented by

$$\alpha(s) = \partial I_{[\,\cdot\, \geq 0]}(s) \quad \text{for } s \geq 0,$$

where $I_{[\,\cdot\, \geq 0]}$ denotes the indicator function over the set $[s \geq 0] := \{s \in \mathbb{R} : s \geq 0\}$ and ∂ means the *subdifferential* in the sense of convex analysis (see, e.g., [18] and also (41) below with $H = \mathbb{R}$). Then (6) can be reformulated as a *doubly nonlinear*-type PDE,

$$\partial_t u + \alpha(\partial_t u) \ni \Delta u + f \quad \text{in } Q. \tag{40}$$

We next reduce the PDE (40) to an *evolution equation*. To this end, define a functional $\phi : L^2(\Omega) \to [0, \infty]$ by

$$\phi(v) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx & \text{if } v \in V, \\ +\infty & \text{if } v \in L^2(\Omega) \setminus V \end{cases}$$

with the *effective domain* $D(\phi) := \{v \in L^2(\Omega) : \phi(v) < +\infty\} = V$. Then we observe that:

Lemma 3.1. *The functional ϕ is convex and lower semicontinuous in $L^2(\Omega)$. In particular, if (12) is satisfied, then the subdifferential operator $\partial\phi$ of ϕ (in $L^2(\Omega)$) is characterized as*

$$D(\partial\phi) = X \cap V, \quad \partial\phi(v) = -\Delta v \quad \textit{for } v \in X \cap V,$$

where Δ stands for the Laplace operator from $D(\Delta) = X \cap V$ into $L^2(\Omega)$ as in Proposition 2.8.

Here let us recall the definition of the *subdifferential operator* $\partial\varphi : H \to H$ of a proper, lower semicontinuous and convex functional φ defined on a Hilbert space H,

$$\partial\varphi(u) := \{\xi \in H : \varphi(v) - \varphi(u) \geq (\xi, v - u)_H \text{ for all } v \in D(\varphi)\} \quad \text{for } u \in D(\varphi), \tag{41}$$

where $(\cdot, \cdot)_H$ stands for the inner product in H and $D(\varphi) := \{w \in H : \varphi(w) < +\infty\}$, with domain $D(\partial\varphi) := \{w \in D(\varphi) : \partial\varphi(w) \neq \emptyset\}$. It is well known that $\partial\varphi$ is a (possibly multivalued) maximal monotone operator in H (see, e.g., [18] for more details).

Proof of Lemma 3.1. We note that the restriction $\phi_0 := \phi|_V$ of ϕ onto V is Fréchet differentiable and the derivative ϕ_0' of ϕ_0 satisfies

$$\langle \phi_0'(u), z \rangle_V = \int_\Omega \nabla u \cdot \nabla z \, dx \quad \text{for all } z \in V. \tag{42}$$

Now, let $u \in D(\partial\phi)$ and $\xi \in \partial\phi(u) \subset L^2(\Omega)$. From the definition of subdifferentials, we find that $\partial\phi(u) \subset \partial\phi_0(u) = \{\phi_0'(u)\}$ for all $u \in D(\partial\phi) \subset D(\phi) = V$. Here $\partial_V \phi_0 : V \to V'$ stands for the subdifferential operator of the functional ϕ_0 and it coincides with the gradient operator $\phi_0' : V \to V'$ of ϕ_0. Hence $\xi = \phi_0'(u)$, i.e., $\partial\phi(u) = \{\phi_0'(u)\}$ and $\phi_0'(u) \in L^2(\Omega)$; here and henceforth, we simply write $\partial\phi(u) = \phi_0'(u)$. It follows that

$$A_1 u = u + \phi_0'(u) = u + \xi \in L^2(\Omega).$$

Moreover, by (12) along with Proposition 2.8, we deduce that $u = A_1^{-1}(u + \xi) \in X \cap V$ and that $u + \xi = A_1 u = -\Delta u + u$. Therefore we deduce that $\partial\phi(u) = \phi_0'(u) = -\Delta u$ and $D(\partial\phi) \subset X \cap V$. On the other hand, it is clear that $X \cap V \subset D(\partial\phi)$, and hence, $D(\partial\phi) = X \cap V$. □

Therefore the initial-boundary value problem for (40) equipped with (7)–(9) can be rewritten as the Cauchy problem for an evolution equation in $L^2(\Omega)$ of $u(t) := u(\cdot, t)$,

$$\partial_t u(t) + \partial I_{[\,\cdot\,\geq 0]}(\partial_t u(t)) + \partial\phi(u(t)) \ni f(t) \ \text{ in } L^2(\Omega), \quad 0 < t < T, \quad u(0) = u_0, \tag{43}$$

where $f(t) := f(\cdot, t)$ and $\partial I_{[\,\cdot\,\geq 0]}$ denotes the subdifferential operator in $L^2(\Omega)$ of the functional $I_{[\,\cdot\,\geq 0]} : L^2(\Omega) \to [0, \infty]$ defined by

$$I_{[\,\cdot\,\geq 0]}(v) = \begin{cases} 0 & \text{if } v \geq 0 \ \text{ a.e. in } \Omega, \\ \infty & \text{otherwise} \end{cases} \qquad \text{for } \ v \in L^2(\Omega).$$

We note that $\partial I_{[\,\cdot\,\geq 0]}(v) = \alpha(v(\cdot))$ for $v \in L^2(\Omega)$, where $\alpha(\cdot)$ is a multivalued function given by (39), and $D(\partial I_{[\,\cdot\,\geq 0]}) = \{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$ (see, e.g., [18]).

Here and henceforth, for simplicity, we use the same notation $I_{[\,\cdot\,\geq 0]}$ for the indicator function over $[0, +\infty)$ defined on $\mathbb{R}$ as well as for that over the set $\{v \in L^2(\Omega) : v \geq 0 \ \text{ a.e. in } \Omega\}$ defined on $L^2(\Omega)$, unless any confusion may arise. Moreover, the subdifferential operators of the both indicator functions are also denoted by $\partial I_{[\,\cdot\,\geq 0]}$.

Strong solutions of (43) are defined as follows:

Definition 3.2 *(Strong solution of* (43)*).* For given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, a function $u \in C([0, T]; L^2(\Omega))$ is called a *strong solution* of (43) on $[0, T]$, if the following conditions are satisfied:

- $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; X \cap V)$;
- It holds that

$$\partial_t u(t) + \partial I_{[\,\cdot\,\geq 0]}(\partial_t u(t)) + \partial\phi(u(t)) \ni f(t) \ \text{ in } L^2(\Omega) \ \text{ for a.e. } t \in (0, T); \tag{44}$$

- $u(0) = u_0$,

where the functionals $I_{[\,\cdot\,\geq 0]}$ and ϕ on $L^2(\Omega)$ are defined as above.

Proposition 3.3 *(Equivalence of solutions). The notion of strong solutions for* (43) *is equivalent to that for* (6)–(9) *defined by Definition* 1.4.

Proof. Since $\alpha(s)$ is the inverse mapping of $s \mapsto (s)_+$, one observes that (1) is equivalent to (40) at each $(x,t) \in Q$. Moreover, due to Lemma 3.1, for each strong solution u of (43), $u(x,t)$ satisfies (40) a.e. in $\Omega \times (0,T)$. Conversely, let u be a strong solution of (6)–(9) in the sense of Definition 1.4. Then from the regularity condition (ii) of Definition 1.4, the right-hand-side of the inclusion

$$\alpha(\partial_t u) \ni \Delta u + f - \partial_t u$$

belongs to $L^2(\Omega)$ for a.e. $t \in (0,T)$. Moreover, recalling that

$$\partial I_{[\,\cdot\,\geq 0]}(v) = \{\xi(\cdot) \in L^2(\Omega) \colon \xi(x) \in \alpha(v(x)) \text{ for a.e. } x \in \Omega\}$$

(see above), the evolution equation (44) holds in $L^2(\Omega)$ for a.e. $t \in (0,T)$. $\square$

We next provide a chain-rule for the function $t \mapsto \phi(u(t))$, which is derived from a standard theory on subdifferential calculus and which will be used frequently to derive energy estimates in later sections.

Lemma 3.4. *We suppose that* $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;X \cap V)$. *Then we have*:

(i) *the function*

$$t \mapsto \phi(u(t)) = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 \, dx$$

belongs to $W^{1,1}(0,T)$;

(ii) *for a.e.* $t \in (0,T)$, *it holds that*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \, dx = \frac{d}{dt}\phi(u(t)) = (\partial\phi(u(t)), \partial_t u(t))_{L^2(\Omega)} = -\int_\Omega \partial_t u \, \Delta u \, dx,$$

where $(\cdot,\cdot)_{L^2(\Omega)}$ *denotes the inner product of* $L^2(\Omega)$;

(iii) $u \in C([0,T];V)$.

Proof. Thanks to Lemma 3.3 of [18], the assertions (i) and (ii) follow immediately. Concerning (iii), since u belongs to $L^\infty(0,T;V)$ and $C([0,T];L^2(\Omega))$, by exploiting Lemma 8.1 of [44], one finds that u is continuous on $[0,T]$ with respect to the weak topology of V. On the other hand, $t \mapsto \|u(t)\|_V$ is continuous on $[0,T]$ by (i). Therefore from the uniform convexity of $\|\cdot\|_V$, we deduce that $t \mapsto u(t)$ is continuous on $[0,T]$ with respect to the strong topology of V. $\square$

Before proceeding to a proof of Theorem 1.6, let us note that

$$|a_+ - b_+|^2 \leq |a_+ - b_+||a-b| = (a_+ - b_+)(a-b) \quad \text{for all } a,b \in \mathbb{R}, \tag{45}$$

since the function $s \mapsto s_+ = s \vee 0$ is nondecreasing and non-expansive, that is, $|a_+ - b_+| \leq |a-b|$. Now, we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. For each $i = 1, 2$, let u_i be a strong solution of (6)–(9) with $u_0 = u_{0,i} \in V$ and $f = f_i \in L^2(0, T; L^2(\Omega))$ and set $u = u_1 - u_2$. Then due to Lemma 3.4, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2\,\mathrm{d}x &= -\int_\Omega \partial_t u\,\Delta u\,\mathrm{d}x \\
&= -\int_\Omega \left[(\Delta u_1 + f_1)_+ - (\Delta u_2 + f_2)_+\right] \\
&\quad \times [(\Delta u_1 + f_1) - (\Delta u_2 + f_2) - f_1 + f_2]\,\mathrm{d}x \\
&\overset{(45)}{\le} -\frac{1}{2}\int_\Omega |(\Delta u_1 + f_1)_+ - (\Delta u_2 + f_2)_+|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |f_1 - f_2|^2\,\mathrm{d}x \quad \text{for a.e. } t \in (0, T),
\end{aligned}
$$

which implies that

$$
\int_\Omega |\partial_t u_1 - \partial_t u_2|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u_1 - \nabla u_2|^2\,\mathrm{d}x \le \int_\Omega |f_1 - f_2|^2\,\mathrm{d}x \quad \text{for a.e. } t \in (0, T).
$$

Integrate both sides with respect to t to obtain

$$
\begin{aligned}
&\int_0^T \|\partial_t u_1(t) - \partial_t u_2(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t + \sup_{t\in[0,T]} \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}^2 \\
&\qquad \le 2\left(\|\nabla u_{0,1} - \nabla u_{0,2}\|_{L^2(\Omega)}^2 + \int_0^T \|f_1(t) - f_2(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t \right). \qquad (46)
\end{aligned}
$$

In particular, if $u_{0,1} = u_{0,2}$ and $f_1 = f_2$, then u_1 coincides with u_2 a.e. in Q_T. Consequently, the solution of (6)–(9) is unique. □

Corollary 3.5 *(Continuous dependence of solutions on data). For each $T > 0$ and $i = 1, 2$, let u_i be the strong solution of* (6)–(9) *on* $[0, T]$ *with $u_0 = u_{0,i} \in V$ and $f = f_i \in L^2(Q_T)$. Then* (46) *holds true.*

4. Existence of solutions and comparison principle

In this section, we shall prove Theorem 1.8 on the existence of solutions for (6)–(9). Let $T > 0$ be fixed. We denote by τ a division $\{t_0, t_1, \ldots, t_m\}$ of the interval $[0, T]$ given by

$$
0 = t_0 < t_1 < \ldots < t_m = T, \quad \tau_k := t_k - t_{k-1} \text{ for } k = 1, \ldots, m, \quad |\tau| := \max_{k=1,\ldots,m} \tau_k.
$$

We shall construct $u_k \in X \cap V$ (for $k = 1, 2, \dots, m$), which is an approximation of $u(t_k)$ for a solution u of (1) by the backward-Euler scheme

$$\frac{u_k - u_{k-1}}{\tau_k} = (\Delta u_k + f_k)_+ \quad \text{a.e. in } \Omega, \tag{47}$$

where $f_k \in L^2(\Omega)$ is given by

$$f_k := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} f(\cdot, s)\, \mathrm{d}s.$$

For given $u_0 \in V$, we shall inductively define $u_k \in V$ for $k = 1, 2, \dots, m$ as a (global) minimizer of the functional

$$J_k(v) := \frac{1}{2\tau_k} \int_\Omega |v|^2\, \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla v|^2\, \mathrm{d}x - \left\langle \frac{u_{k-1}}{\tau_k} + f_k, v \right\rangle_V \quad \text{for } v \in V \tag{48}$$

subject to

$$v \in K_0^k := \{v \in V : v \ge u_{k-1} \text{ a.e. in } \Omega\}. \tag{49}$$

Remark 4.1 *(Derivation of the discretized problems).* The minimization problems with constraints stated above can be also derived from a discretization of the evolution equation (43), which is equivalent to (6) (see Section 3). A natural time-discretization of (43) may be given as

$$\frac{u_k - u_{k-1}}{\tau_k} + \partial_V I_{[\,\cdot \ge 0]} \left(\frac{u_k - u_{k-1}}{\tau_k} \right) - \Delta u_k \ni f_k \quad \text{in } V' \tag{50}$$

(here ∂_V stands for the subdifferential of the functional $I_{[\,\cdot \ge 0]}$ restricted onto V), which is an Euler–Lagrange equation for the functional

$$E_k(v) := \frac{1}{2\tau_k} \int_\Omega |v|^2\, \mathrm{d}x + I_{[\,\cdot \ge 0]} \left(\frac{v - u_{k-1}}{\tau_k} \right) + \frac{1}{2} \int_\Omega |\nabla v|^2\, \mathrm{d}x - \left\langle \frac{u_{k-1}}{\tau_k} + f_k, v \right\rangle_V \quad \text{for } v \in V.$$

Indeed, since E_k is coercive, lower semicontinuous and convex in V, E_k admits a global minimizer u_k over V, and moreover, u_k solves (50) in V'. Here we note that the minimization of E_k over V is equivalent to that of J_k over K_0^k from the fact that

$$I_{[\,\cdot \ge 0]} \left(\frac{v - u_{k-1}}{\tau_k} \right) = I_{[\,\cdot \ge u_{k-1}]}(v) := \begin{cases} 0 & \text{if } v \ge u_{k-1} \text{ a.e. in } \Omega, \\ \infty & \text{otherwise} \end{cases} \quad \text{for } v \in L^2(\Omega).$$

Applying the regularity theory established in Section 2, one can actually obtain the unique minimizer u_k of J_k over K_0^k for each k. More precisely, we obtain the following theorem, where we set

$$g_k := \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k. \tag{51}$$

Lemma 4.2 *(Existence and regularity of minimizers). Suppose that (12) is satisfied and that*

$$u_0 \in V, \quad A_0 u_0 \in \mathcal{M}(\overline{\Omega}), \quad (A_0 u_0)_+ \in L^2(\Omega). \tag{52}$$

For each $k = 1, 2, \ldots, m$, there exists a unique element $u_k \in K_0^k$ which minimizes (48) subject to (49). Moreover, for each $k = 1, 2, \ldots, m$, the minimizer u_k belongs to X and fulfills (47), that is,

$$u_k - u_{k-1} \geq 0 \ \textit{ a.e. in } \ \Omega, \tag{53}$$

$$g_k \geq 0 \ \textit{ a.e. in } \ \Omega, \tag{54}$$

$$\langle g_k, u_k - u_{k-1} \rangle_V = 0. \tag{55}$$

Furthermore, one has

$$\langle g_k, v - u_k \rangle_V \geq 0 \quad \textit{for all } \ v \in K_0^k, \tag{56}$$

$$\langle g_k + f_k - A_0 u_{k-1}, v - u_k \rangle_V \geq 0 \quad \textit{for all } \ v \in K_1^k, \tag{57}$$

where the set K_1^k is given by

$$K_1^k := \left\{ v \in V : \frac{v - u_{k-1}}{\tau_k} + A_0 v - f_k \geq 0 \text{ in } V' \right\}.$$

Moreover, it holds that

$$0 \leq g_k \leq (A_0 u_{k-1} - f_k)_+ \ \textit{ a.e. in } \Omega \ \textit{ for each } \ k = 1, \ldots, m. \tag{58}$$

Proof. Denote by Δ the Laplace operator from $X \cap V$ into $L^2(\Omega)$ (see Section 3). Then $A_0 w = -\Delta w$ for $w \in X \cap V$, since $\partial_\nu w = 0$ on Γ_N. Here and henceforth, if no confusion may arise, we also simply write $-\Delta u_0$ instead of $A_0 u_0 \in V'$, although u_0 may not belong to X. Let us start with $k = 1$. Set $\sigma = 1/\tau_k > 0$ (i.e., $Au := A_\sigma u = u/\tau_k - \Delta u$), $f = f_k + u_{k-1}/\tau_k \in L^2(\Omega)$ and $\psi = u_{k-1}$. Here by (52) and Lemma A.1, we find that $(Au_0 - f)_+ = (A_0 u_0 - f_1)_+ \in L^2(\Omega)$. Then f and ψ satisfy (23) and (24) by (52). One can write $K_0^k = K_0$ and $J_k(v) = J(v)$ for $v \in V$ with K_0 and $J(v)$ defined by (18) and (19) along with (16). Hence, one can apply Proposition 2.3 and Theorem 2.4 to the minimization problem of J_k over K_0^k. Then the minimizer $u_k \in K_0^k$ of J_k over K_0^k uniquely exists, and furthermore, (54)–(57) follow immediately from the fact $u_k \in K_1^k$, (b), (c) and (d) of Proposition 2.3, respectively. Moreover, by virtue of (12) and Theorem 2.1, one can deduce that $u_k \in X$. Repeating the argument above for $k = 2, 3, \ldots, m$, we can inductively obtain $u_k \in K_0^k \cap X$ satisfying (54)–(57) for each $k = 2, 3, \ldots, m$.

Finally, by Theorem 2.1, we can assure that

$$f_k + \frac{u_{k-1}}{\tau_k} \leq \frac{u_k}{\tau_k} - \Delta u_k \leq \left(f_k + \frac{u_{k-1}}{\tau_k} \right) \vee \left(\frac{u_{k-1}}{\tau_k} - \Delta u_{k-1} \right) \qquad \text{for a.e. } x \in \Omega,$$

which is equivalent to (58). Here we also remark that the right-hand-side above belongs to $L^2(\Omega)$ by (52) and Lemma A.1. □

Proof of Theorem 1.8. Let us define the *piecewise linear interpolant* $u_\tau \in W^{1,\infty}(0,T;V) \cap W^{1,\infty}(\tau_1,T;X)$ of $\{u_k\}$ and the *piecewise constant interpolants* $\bar{u}_\tau \in L^\infty(0,T;X \cap V)$ and $\bar{f}_\tau \in L^\infty(0,T;L^2(\Omega))$ of $\{u_k\}$ and $\{f_k\}$, respectively, by

$$u_\tau(t) := u_{k-1} + \frac{t-t_{k-1}}{\tau_k}(u_k - u_{k-1}) \quad \text{for } t \in [t_{k-1},t_k] \text{ and } k = 1,\dots,m,$$

$$\bar{u}_\tau(t) := u_k, \quad \bar{f}_\tau(t) := f_k \quad \text{for } t \in (t_{k-1},t_k] \text{ and } k = 1,\dots,m.$$

By summing up (55) for $k = 1,\dots,\ell$ with an arbitrary natural number $\ell \le m$, we have

$$\begin{aligned} &\sum_{k=1}^{\ell} \tau_k \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega |\nabla u_\ell|^2 \,dx - \frac{1}{2}\int_\Omega |\nabla u_0|^2 \,dx \\ &\quad \le \sum_{k=1}^{\ell} \tau_k \left(f_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)} \le \frac{1}{2}\sum_{k=1}^{\ell} \tau_k \|f_k\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{k=1}^{\ell} \tau_k \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2, \end{aligned} \tag{59}$$

which implies

$$\int_0^t \|\partial_t u_\tau(s)\|_{L^2(\Omega)}^2 \,ds + \|\nabla \bar{u}_\tau(t)\|_{L^2(\Omega)}^2 \le \|\nabla u_0\|_{L^2(\Omega)}^2 + \int_0^T \|\bar{f}_\tau(s)\|_{L^2(\Omega)}^2 \,ds \quad \text{for all } t \in [0,T].$$

Hence, we obtain

$$\begin{aligned} &\|\partial_t u_\tau\|_{L^2(0,T;L^2(\Omega))}^2 + \sup_{t\in[0,T]} \|\nabla \bar{u}_\tau(t)\|_{L^2(\Omega)}^2 + \sup_{t\in[0,T]} \|\nabla u_\tau(t)\|_{L^2(\Omega)}^2 \\ &\qquad \le C\left(\|\nabla u_0\|_{L^2(\Omega)}^2 + \|\bar{f}_\tau\|_{L^2(0,T;L^2(\Omega))}^2 \right). \end{aligned} \tag{60}$$

Now, let us take a limit as $m \to \infty$ such that $|\tau| \to 0$ and note that

$$\bar{f}_\tau \to f \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{61}$$

In particular, $\{\bar{f}_\tau\}$ is bounded in $L^2(0,T;L^2(\Omega))$. Indeed, one can verify that

$$\|\bar{f}_\tau\|_{L^2(0,T;L^2(\Omega))} \le \|f\|_{L^2(0,T;L^2(\Omega))}.$$

From the uniform estimate (60), one can take a function $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$ (in particular, $u \in C([0,T];L^2(\Omega))$ as well) such that, up to a (non-relabeled) subsequence,

$$u_\tau \to u \qquad \text{weakly in } W^{1,2}(0,T;L^2(\Omega)), \tag{62}$$

$$\text{weakly star in } L^\infty(0,T;V), \tag{63}$$

$$\text{strongly in } C([0,T];L^2(\Omega)), \tag{64}$$

$$\bar{u}_\tau \to u \qquad \text{weakly star in } L^\infty(0,T;V), \tag{65}$$

$$u_\tau(T) \to u(T) \qquad \text{weakly in } V. \tag{66}$$

Here, the weak and weak star convergence of u_τ and $\bar{u}_\tau$ immediately follow from the uniform estimate (60). Moreover, we also note that u_τ and $\bar{u}_\tau$ possess a common limit function. Indeed, by a simple calculation, we observe that

$$\begin{aligned}
\|u_\tau(t) - \bar{u}_\tau(t)\|_{L^2(\Omega)} &= \left|\frac{t_k - t}{\tau_k}\right| \|u_k - u_{k-1}\|_{L^2(\Omega)} \\
&\leq \left\|\frac{u_k - u_{k-1}}{\tau_k}\right\|_{L^2(\Omega)} \tau_k \\
&\overset{(59)}{\leq} C|\tau|^{1/2} \quad \text{for all } t \in (t_{k-1}, t_k],\ k = 1, 2, \ldots, m,
\end{aligned}$$

which yields that

$$\sup_{t\in[0,T]} \|u_\tau(t) - \bar{u}_\tau(t)\|_{L^2(\Omega)} \leq C|\tau|^{1/2} \to 0.$$

Thus u_τ and $\bar{u}_\tau$ (weakly) converge to a common limit function. Furthermore, since V is compactly embedded in $L^2(\Omega)$, due to Ascoli's compactness lemma along with (60), we obtain the strong convergence (64). Since $u_\tau(T) = u_m$ is bounded in V by (59), one can also derive (66) from (64). We further observe that $u(0) = u_0$.

We next estimate $\Delta\bar{u}_\tau$ in $L^2(0,T;L^2(\Omega))$ by using (58) and the assumption (14). We first rewrite (58) as

$$-\frac{u_k - u_{k-1}}{\tau_k} + f_k \leq -\Delta u_k \leq \left(-\frac{u_k - u_{k-1}}{\tau_k} + f_k\right) \vee \left(-\frac{u_k - u_{k-1}}{\tau_k} - \Delta u_{k-1}\right) \quad \text{a.e. in } \Omega \tag{67}$$

for $k = 2, 3, \ldots, m$. Since $(u_k - u_{k-1})/\tau_k \geq 0$ a.e. in Ω by $u_k \in K_0^k$, we observe by (14) that

$$\text{(The right-hand side of (67))} \leq f_k \vee (-\Delta u_{k-1}) \leq f^* \vee (-\Delta u_{k-1}) \quad \text{a.e. in } \Omega,$$

which also iteratively implies that

$$\begin{aligned}
-\Delta u_k &\leq f^* \vee (-\Delta u_{k-1}) \\
&\leq f^* \vee \big(f^* \vee (-\Delta u_{k-2})\big) \\
&= f^* \vee (-\Delta u_{k-2}) \leq \cdots \leq f^* \vee (-\Delta u_1) \quad \text{a.e. in } \Omega
\end{aligned}$$

for $k = 2, 3, \ldots, m$. Here by Lemma A.1 we find that

$$
\begin{aligned}
f^* \vee (-\Delta u_1) &\overset{(58)}{\leq} f^* \vee \left\{ (-\Delta u_0 - f_1)_+ - \frac{u_1 - u_0}{\tau_1} + f_1 \right\} \\
&\leq f^* \vee \{ (-\Delta u_0 - f_1)_+ + f_1 \} \\
&\leq f^* \vee \{ (-\Delta u_0)_+ + (-f_1)_+ + f_1 \} \\
&= f^* \vee \{ (-\Delta u_0)_+ + (f_1)_+ \} \\
&\leq (-\Delta u_0)_+ + (f^*)_+.
\end{aligned}
$$

Thus we obtain

$$
-\frac{u_k - u_{k-1}}{\tau_k} + f_k \leq -\Delta u_k \leq (-\Delta u_0)_+ + (f^*)_+ \quad \text{a.e. in } \Omega,
$$

which together with (52) yields that

$$
\|\Delta u_k\|^2_{L^2(\Omega)} \leq 2 \left(\|(-\Delta u_0)_+\|^2_{L^2(\Omega)} + \|f^*\|^2_{L^2(\Omega)} + \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|^2_{L^2(\Omega)} + \|f_k\|^2_{L^2(\Omega)} \right)
$$

for $k = 1, 2, \ldots, m$. Hence we deduce that

$$
\begin{aligned}
\int_0^T \|\Delta \bar{u}_\tau(t)\|^2_{L^2(\Omega)} \, \mathrm{d}t &\leq CT \left(\|f^*\|^2_{L^2(\Omega)} + 1 \right) \\
&\quad + C \int_0^T \|\partial_t u_\tau(t)\|^2_{L^2(\Omega)} \, \mathrm{d}t + C \int_0^T \left\| \bar{f}_\tau(t) \right\|^2_{L^2(\Omega)} \, \mathrm{d}t \leq C \qquad (68)
\end{aligned}
$$

by using (60) and (61).

Exploiting Proposition 2.8 with (12), we see that $(I - \Delta) \in \mathrm{Isom}(X \cap V, L^2(\Omega))$, which together with (68) gives

$$
\int_0^T \|\bar{u}_\tau(t)\|^2_X \, \mathrm{d}t \leq C \int_0^T \left(\|\Delta \bar{u}_\tau(t)\|^2_{L^2(\Omega)} + \|\bar{u}_\tau(t)\|^2_{L^2(\Omega)} \right) \mathrm{d}t \leq C.
$$

Therefore we have, up to a (non-relabeled)subsequence,

$$
\begin{aligned}
\bar{u}_\tau &\to u && \text{weakly in } L^2(0, T; X), \\
\Delta \bar{u}_\tau &\to \Delta u && \text{weakly in } L^2(0, T; L^2(\Omega)),
\end{aligned}
$$

which particularly implies $u(t) \in D(\Delta) = X \cap V$ for a.e. $t \in (0, T)$. Therefore the piecewise constant interpolant $\bar{g}_\tau$ of $\{g_k\}$ defined by

$$
\bar{g}_\tau(t) := g_k \overset{(51)}{=} \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k \quad \text{for } t \in (t_{k-1}, t_k]
$$

converges to

$$\partial_t u - \Delta u - f =: g \tag{69}$$

weakly in $L^2(0, T; L^2(\Omega))$.

It remains to prove that u solves (6) for a.e. $(x, t) \in Q_T$. To this end, we recall the evolution equation (43) equivalent to (6). Then it suffices to check that

$$\partial_t u \geq 0 \text{ a.e. in } Q_T \quad \text{and} \quad -g(t) \in \partial I_{[\,\cdot\,\geq 0]}(\partial_t u(t)) \text{ for a.e. } t \in (0, T).$$

To this end, we employ the so-called Minty's trick for maximal monotone operators, since $\partial I_{[\,\cdot\,\geq 0]}$ is maximal monotone in $L^2(\Omega)$.

Proposition 4.3 *(Demiclosedness of maximal monotone operators (see, e.g., [18,21,11])). Let $A : H \to H$ be a (possibly multivalued) maximal monotone operator defined on a Hilbert space H equipped with a inner product $(\cdot, \cdot)_H$. Let $[u_n, \xi_n]$ be in the graph of A such that $u_n \to u$ weakly in H and $\xi_n \to \xi$ weakly in H. Suppose that*

$$\limsup_{n \to +\infty} (\xi_n, u_n)_H \leq (\xi, u)_H.$$

Then $[u, \xi]$ belongs to the graph of A, and moreover, it holds that

$$\lim_{n \to +\infty} (\xi_n, u_n)_H = (\xi, u)_H.$$

Note that $(u_k - u_{k-1})/\tau_k \geq 0$ a.e. in Ω. For an arbitrary $w \in D(I_{[\,\cdot\,\geq 0]}) = \{v \in L^2(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$, substitute $v = w\tau_k + u_{k-1} \in K_0^k$ to (56). Then we see that

$$0 \overset{(56)}{\geq} \langle -g_k, v - u_k \rangle_V = \tau_k \left(-g_k, w - \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)},$$

which together with the arbitrariness of $w \in D(I_{[\,\cdot\,\geq 0]})$ and the definition of $I_{[\,\cdot\,\geq 0]}$ implies that

$$-g_k \in \partial I_{[\,\cdot\,\geq 0]} \left(\frac{u_k - u_{k-1}}{\tau_k} \right), \qquad \text{i.e., } -\bar{g}_\tau(t) \in \partial I_{[\,\cdot\,\geq 0]} \left(\partial_t u_\tau(t) \right).$$

Moreover, for $k = 1, 2, \ldots, m$, we find by (51) that

$$\left(-g_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)} \leq - \left\| \frac{u_k - u_{k-1}}{\tau_k} \right\|_{L^2(\Omega)}^2 - \frac{\phi(u_k) - \phi(u_{k-1})}{\tau_k} + \left(f_k, \frac{u_k - u_{k-1}}{\tau_k} \right)_{L^2(\Omega)},$$

which leads us to get

$$\int_0^T (-\bar{g}_\tau(t), \partial_t u_\tau(t))_{L^2(\Omega)}\,\mathrm{d}t \le -\int_0^T \|\partial_t u_\tau(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t - \phi(u_\tau(T)) + \phi(u_0) + \int_0^T \big(\bar{f}_\tau(t), \partial_t u_\tau(t)\big)_{L^2(\Omega)}\,\mathrm{d}t.$$

Taking a limsup as $|\tau| \to 0$ in both sides, exploiting the weak lower semicontinuity of norms and the functional $\phi(\cdot)$, and recalling Lemma 3.4, we conclude that

$$\begin{aligned}\limsup_{|\tau|\to 0} \int_0^T (-\bar{g}_\tau(t), \partial_t u_\tau(t))_{L^2(\Omega)}\,\mathrm{d}t &\le -\int_0^T \|\partial_t u(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t - \phi(u(T)) + \phi(u_0) \\ &\quad + \int_0^T (f(t), \partial_t u(t))_{L^2(\Omega)}\,\mathrm{d}t \\ &= \int_0^T (-\partial_t u(t) + \Delta u(t) + f(t), \partial_t u(t))_{L^2(\Omega)}\,\mathrm{d}t \\ &\overset{(69)}{=} \int_0^T (-g(t), \partial_t u(t))_{L^2(\Omega)}\,\mathrm{d}t. \end{aligned} \tag{70}$$

Consequently, by virtue of the (weak) closedness of maximal monotone operators (see Proposition 4.3 above), it follows that $\partial_t u(t) \in D(\partial I_{[\,\cdot\,\ge 0]})$, i.e., $\partial_t u(t) \ge 0$ a.e. in Ω, and $-g(t) \in \partial I_{[\,\cdot\,\ge 0]}(\partial_t u(t))$ for a.e. $t \in (0,T)$. Therefore u solves (43), and hence, u is a strong solution of (6)–(9). Thus Theorem 1.8 has been proved. □

Remark 4.4. To prove that u is a strong solution of (6)–(9), it is possible to show the conditions (V1)–(V6) of Theorem 6.1, instead of the last argument of the proof of Theorem 1.8. Actually, (V2) and (V3) directly follow from (53) and (54) by taking limit of $|\tau| \to 0$, respectively. The condition (V4) follows from the estimate (70), since the left-hand side of (70) is zero and the right-hand side is non-positive.

Due to Theorem 1.6, the limit of $\{u_\tau\}$ and $\{\bar{u}_\tau\}$ is unique, whence they converge along the full sequence.

Corollary 4.5. *Sequences $\{u_\tau\}$ and $\{\bar{u}_\tau\}$ converge to the unique solution u of* (6)–(9) *as $|\tau| \to 0_+$.*

We next prove Theorem 1.11.

Proof of Theorem 1.11. Let u^1 and u^2 be strong solutions of (6)–(9) with $u_0 = u_0^i$ and $f = f^i$ for $i = 1, 2$, respectively. By the uniqueness of solutions (see Theorem 1.6) and the construction of solutions discussed so far, one can take discretized solutions $\{u_k^i\}$ for $i = 1, 2$ such that the piecewise linear interpolant u_τ^i of $\{u_k^i\}$ converges to u^i strongly in $C([0, T]; L^2(\Omega))$ as $|\tau| \to 0$, and they solve the variational inequalities

$$\begin{cases} u_k^i \in K_0(u_{k-1}^i) := \{v \in V : v \geq u_{k-1}^i \text{ a.e. in } \Omega\}, \\ a_k(u_k^i, v - u_k^i) \geq \int_\Omega \left(f_k^i + u_{k-1}^i/\tau_k\right)(v - u_k^i)\,\mathrm{d}x \quad \text{for all } v \in K_0(u_{k-1}^i), \end{cases}$$

where $a_k(\cdot,\cdot)$ stands for the bilinear form given by

$$a_k(u, v) = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x + \frac{1}{\tau_k}\int_\Omega uv\,\mathrm{d}x \quad \text{for } u, v \in V.$$

By iteratively applying the comparison theorem for elliptic variational inequalities (see Theorem 2.2), from the fact that $f^1 \leq f^2$ a.e. in Q_T and $u_0^1 \leq u_0^2$ a.e. in Ω, one can deduce that

$$u_k^1 \leq u_k^2 \text{ a.e. in } Q_T \quad \text{for all } k = 1, 2, \ldots, m,$$

which also implies $u_\tau^1(t) \leq u_\tau^2(t)$ a.e. in Ω for all $t \in (0, T)$. Then passing to the limit as $|\tau| \to 0$, we conclude that $u^1 \leq u^2$ a.e. in Q_T. □

5. Long-time behavior of solutions

This section is devoted to proving Theorem 1.12. Let us begin with deriving a uniform estimate for $u(t)$ for $t \geq 0$. To do so, recall the construction of the unique solution $u = u(x, t)$ of (6)–(9) performed in the proof of Theorem 1.8 and particularly note that

$$u_k \geq u_{k-1} \quad \text{and} \quad g_k := \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k - f_k \geq 0 \quad \text{a.e. in } \Omega. \tag{71}$$

It follows that

$$h_k := g_k + f_k = \frac{u_k - u_{k-1}}{\tau_k} - \Delta u_k \geq -\Delta u_k \quad \text{a.e. in } \Omega. \tag{72}$$

We also recall the estimate (58), which gives

$$f_k \leq h_k \leq (-\Delta u_{k-1}) \vee f_k \quad \text{a.e. in } \Omega. \tag{73}$$

Therefore by (H3) we find that

$$\begin{aligned} f_k \overset{(73)}{\leq} h_k &\overset{(73)}{\leq} (-\Delta u_{k-1}) \vee f_k \\ &\overset{(72)}{\leq} h_{k-1} \vee f^* \\ &\leq (h_{k-2} \vee f^*) \vee f^* \\ &= h_{k-2} \vee f^* \leq \cdots \leq h_1 \vee f^* \overset{(73)}{\leq} (-\Delta u_0) \vee f^* \quad \text{a.e. in } \Omega, \end{aligned}$$

which together with the assumption that $u_0 \in X \cap V$ gives

$$\|h_k\|_{L^2(\Omega)} \le \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f_k\|_{L^2(\Omega)} \\ \le \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,\infty;L^2(\Omega))}. \tag{74}$$

Here we used the fact that $\|f_k\|_{L^2(\Omega)} \le \|f\|_{L^\infty(0,\infty;L^2(\Omega))}$ for all k. Moreover, set

$$\bar{h}_\tau(t) := \partial_t u_\tau(t) - \Delta \bar{u}_\tau(t) = h_k \quad \text{for } t \in (t_{k-1}, t_k].$$

Recalling the convergence of approximate solutions obtained in the proof of Theorem 1.8, we observe that

$$\bar{h}_\tau \to \partial_t u - \Delta u =: h \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

On the other hand, since $\{\bar{h}_\tau\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by (74), we assure, up to a (non-relabeled) subsequence, that

$$\bar{h}_\tau \to h \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega))$$

as $|\tau| \to 0$. Moreover, from the lower semicontinuity of the L^∞-norm in the weak star topology, we have, by (74),

$$\|h\|_{L^\infty(0,T;L^2(\Omega))} \le \liminf_{\tau\to 0} \|\bar{h}_\tau\|_{L^\infty(0,T;L^2(\Omega))} \le \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,\infty;L^2(\Omega))}$$

for each $T > 0$. Since the bound is independent of $T > 0$, one can derive that

$$\|h\|_{L^\infty(0,\infty;L^2(\Omega))} \le \|\Delta u_0\|_{L^2(\Omega)} + \|f^*\|_{L^2(\Omega)} + \|f\|_{L^\infty(0,\infty;L^2(\Omega))}. \tag{75}$$

Note that $(\xi, v)_{L^2(\Omega)} = 0$ for all $v \in L^2(\Omega)$ with $v \ge 0$ a.e. in Ω and $\xi \in \partial I_{[\,\cdot\,\ge 0]}(v)$. Thus testing (43) by $\partial_t u(t)$, we have

$$\begin{aligned} &\|\partial_t u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2_{L^2(\Omega)} \\ &= (f(t), \partial_t u(t))_{L^2(\Omega)} \\ &= (f(t) - f_\infty, \partial_t u(t))_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}(f_\infty, u(t))_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|f(t) - f_\infty\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}(f_\infty, u(t))_{L^2(\Omega)}. \end{aligned}$$

Define an energy functional E on V by

$$E(v) := \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - (f_\infty, v)_{L^2(\Omega)} \quad \text{for } v \in V.$$

Then one has

$$\frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) \le \frac{1}{2}\|f(t) - f_\infty\|^2_{L^2(\Omega)} \quad \text{for a.e. } t \ge 0, \tag{76}$$

which implies the non-increase of the function

$$t \mapsto E(u(t)) - \frac{1}{2}\int_0^t \|f(\tau) - f_\infty\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \quad \text{for } t \ge 0.$$

Moreover, by using the Poincaré inequality (due to (H1)), we have

$$E(v) \ge \frac{1}{4}\|\nabla v\|^2_{L^2(\Omega)} - C\|f_\infty\|^2_{L^2(\Omega)} \quad \text{for all } v \in V. \tag{77}$$

Thus integrating (76) over $(0, s)$ and using (H2) and (77) we obtain

$$\int_0^\infty \|\partial_t u(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C, \tag{78}$$

$$\sup_{t\ge 0}\|\nabla u(t)\|_{L^2(\Omega)} \le C, \tag{79}$$

which also yields

$$\sup_{t\ge 0}\|\Delta u(t)\|_{V'} \le C. \tag{80}$$

Moreover, by virtue of (75), we have

$$\|g\|_{L^\infty(0,\infty;L^2(\Omega))} \le \|h\|_{L^\infty(0,\infty;L^2(\Omega))} + \|f\|_{L^\infty(0,\infty;L^2(\Omega))} \le M \tag{81}$$

for some constant M. Here we used the fact that $g(t) = h(t) - f(t)$.

Let $I \subset (0, \infty)$ be the set of all $t \ge 0$ for which (43) holds true and $\|g(t)\|_{L^2(\Omega)}$ is bounded by M as in (81). Then the set $(0, \infty) \setminus I$ has zero Lebesgue measure. Recalling by (78) and (H2) that

$$\int_0^\infty \left(\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|f(t) - f_\infty\|^2_{L^2(\Omega)}\right)\mathrm{d}t < \infty,$$

one can take a sequence $s_n \in [n, n+1] \cap I$ such that

$$\partial_t u(s_n) \to 0 \qquad \text{strongly in } L^2(\Omega), \tag{82}$$

$$f(s_n) \to f_\infty \qquad \text{strongly in } L^2(\Omega) \tag{83}$$

as $n \to \infty$.

Moreover, by using the preceding uniform (in t) estimates and the compact embedding $V \hookrightarrow L^2(\Omega)$, we deduce, up to a (non-relabeled) subsequence, that

$$u(s_n) \to z \qquad \text{weakly in } V, \tag{84}$$

$$\text{strongly in } L^2(\Omega), \tag{85}$$

$$-\Delta u(s_n) \to -\Delta z \qquad \text{weakly in } V', \tag{86}$$

$$-g(t) \to \xi \qquad \text{weakly in } L^2(\Omega) \tag{87}$$

with some $z \in V$ and $\xi \in L^2(\Omega)$. From the demiclosedness of $\partial I_{[\,\cdot\,\geq 0]}$ in $L^2(\Omega)$ and the fact that $-g(t) \in \partial I_{[\,\cdot\,\geq 0]}(\partial_t u(t))$ for a.e. $t \in (0,\infty)$, it follows that $\xi \in \partial I_{[\,\cdot\,\geq 0]}(0)$, that is, $\xi \leq 0$ a.e. in Ω by $\partial I_{[\,\cdot\,\geq 0]}(0) = (-\infty, 0]$. Moreover, by (43) and (83), we get $\xi - \Delta z = f_\infty$, which leads us to $f_\infty + \Delta z = \xi \leq 0$ a.e. in Ω. Furthermore, by (12) along with Proposition 2.8, the limit z belongs to X, since $z - \Delta z = z - \xi + f_\infty \in L^2(\Omega)$.

Therefore we derive that

$$\begin{aligned}
\|\nabla u(s_n)\|^2_{L^2(\Omega)} &= (-\Delta u(s_n), u(s_n))_{L^2(\Omega)} \\
&= (-\partial_t u(s_n), u(s_n))_{L^2(\Omega)} + (g(s_n), u(s_n))_{L^2(\Omega)} + (f(s_n), u(s_n))_{L^2(\Omega)} \\
&\to (-\xi + f_\infty, z)_{L^2(\Omega)} = (-\Delta z, z)_{L^2(\Omega)} = \|\nabla z\|^2_{L^2(\Omega)}.
\end{aligned}$$

From the uniform convexity of V, it holds that

$$u(s_n) \to z \qquad \text{strongly in } V. \tag{88}$$

We shall next verify the convergence of the solution $u(t)$ to the same limit z as $t \to \infty$, that is, $u(t_n) \to z$ for any sequence $t_n \to \infty$ and the limit z is independent of the choice of the sequence (t_n). Subtracting the stationary equation

$$\partial I_{[\,\cdot\,\geq 0]}(0) - \Delta z \ni f_\infty$$

from the evolution equation (43), we see that

$$\partial_t u(t) + \partial I_{[\,\cdot\,\geq 0]}(\partial_t u(t)) - \partial I_{[\,\cdot\,\geq 0]}(0) - \Delta\left(u(t) - z\right) \ni f(t) - f_\infty.$$

Test it by $\partial_t u(t)$ to get

$$\frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u(t) - z)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|f(t) - f_\infty\|^2_{L^2(\Omega)}.$$

Integrate both sides over (s_n, τ) for $\tau > s_n$. Then it follows from (88) that

$$\begin{aligned}
\frac{1}{2}\sup_{\tau \geq s_n}\|\nabla(u(\tau) - z)\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\|\nabla(u(s_n) - z)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{s_n}^{\infty}\|f(t) - f_\infty\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&\stackrel{\text{(H2)}}{\to} 0.
\end{aligned}$$

Thus $u(t)$ converges to the limit z strongly in V as $t \to \infty$. This completes the proof of the first half of the assertion.

We next prove the second half of the assertion. In addition, assume that $f(x,t) \le f_\infty(x)$ for a.e. $x \in Q$ and let $\bar{z} \in X \cap V$ be the unique solution of the variational inequality (VI)(u_0, f_∞). Then by Proposition 2.3 and Theorem 2.4 for $A = A_\sigma$ with $\sigma = 0$, $\bar{z}$ satisfies $-\Delta \bar{z} \ge f_\infty$ a.e. in Ω, and moreover, we deduce that $U(x,t) := \bar{z}(x)$ becomes a strong solution of (6) by observing that

$$\partial_t U \equiv 0 \ \text{ and } \ \Delta U(x,t) + f(x,t) \le \Delta \bar{z}(x) + f_\infty(x) \le 0 \ \text{ a.e. in } Q.$$

Hence by the comparison principle for the evolutionary problem (6) (see Theorem 1.11), we assure that $u(x,t) \le \bar{z}(x)$ for a.e. $(x,t) \in Q$. Letting $t \to \infty$ and recalling (85), we obtain

$$z(x) \le \bar{z}(x) \quad \text{ for a.e. } x \in \Omega.$$

On the other hand, since z belongs to $X \cap V$ and satisfies $z \ge u_0$ and $-\Delta z \ge f_\infty$ in V', applying the comparison theorem for variational inequalities of obstacle type (see Lemma 2.11) to (VI)(u_0, f_∞), we assure that $\bar{z} \le z$ a.e. in Ω. Consequently, we conclude that $z = \bar{z}$ a.e. in Ω. Thus we have proved the second half of the assertion of Theorem 1.12. $\square$

6. Other equivalent formulations

In this section, we discuss other formulations of solutions for (6)–(9) equivalent to those defined by Definition 1.4. Let us start with a *complementarity form* of strong solutions.

Theorem 6.1. *Let $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$. Then u is a strong solution of the problem (6)–(9) on $[0,T]$, if and only if the following six conditions are satisfied*:

(V1) $u \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; X \cap V)$,
(V2) $\partial_t u \ge 0$ *a.e. in* Q_T,
(V3) $\partial_t u - \Delta u - f \ge 0$ *a.e. in* Q_T,
(V4) $(\partial_t u - \Delta u - f)\, \partial_t u = 0$ *a.e. in* Q_T,
(V5) $u(0,\cdot) = u_0$.

Proof. If u satisfies (V1) (or (i) of Definition 1.4), one can define the following measurable subsets of Q_T:

$$Q_0 := \{(x,t) \in Q_T : \partial_t u \ne (\Delta u + f)_+\},$$
$$Q_1 := \{(x,t) \in Q_T : \partial_t u = (\Delta u + f)_+ > 0\},$$
$$Q_2 := \{(x,t) \in Q_T : \partial_t u = (\Delta u + f)_+ = 0\},$$

which are disjoint and satisfy $Q_T = Q_0 \cup Q_1 \cup Q_2$.

Let u satisfy (i)–(iii) of Definition 1.4. Conditions (V1) and (V5) follow immediately. From (ii) of Definition 1.4, it follows that

$$\mathcal{H}^{n+1}(Q_0) = 0,$$

and moreover, by definition,

$$\partial_t u > 0, \quad \partial_t u - \Delta u - f = 0 \quad \text{a.e. in } Q_1,$$
$$\partial_t u = 0, \quad \partial_t u - \Delta u - f \geq 0 \quad \text{a.e. in } Q_2.$$

Hence (V2), (V3) and (V4) follows. Consequently, every strong solution u of (6)–(9) in the sense of Definition 1.4 satisfies all the conditions (V1)–(V5).

Conversely, let u satisfy (V1)–(V5). Conditions (i) and (iii) of Definition 1.4 follow from (V1) and (V5). Let us next show that $\mathcal{H}^{n+1}(Q_0) = 0$, which is equivalent to the condition (ii) of Definition 1.4. Define

$$Q^* := \{(x,t) \in Q_T : \partial_t u \geq 0 \text{ and } \partial_t u - \Delta u - f \geq 0 \text{ at } (x,t)\}.$$

Then it holds that $\mathcal{H}^{n+1}(Q_T \setminus Q^*) = 0$ by (V2) and (V3).

We claim that

$$\partial_t u > 0, \quad \partial_t u - \Delta u - f > 0 \quad \text{at each } (x,t) \in Q_0 \cap Q^*. \tag{89}$$

Indeed, by the definitions of Q_0 and Q^*, u satisfies the following conditions at each $(x,t) \in Q_0 \cap Q^*$:

$$\partial_t u \neq (\Delta u + f)_+, \tag{90}$$
$$\partial_t u \geq 0, \tag{91}$$
$$\partial_t u - \Delta u - f \geq 0. \tag{92}$$

If $\partial_t u = 0$ at some $(x_0, t_0) \in Q_0 \cap Q^*$, then $\Delta u + f > 0$ at (x_0, t_0) by (90) and it contradicts (92). Hence, we obtain $\partial_t u > 0$ at each point of $Q_0 \cap Q^*$ by (91). Similarly, if $\partial_t u - \Delta u - f = 0$ at some $(x_0, y_0) \in Q_0 \cap Q^*$, then $0 < \partial_t u = \Delta u + f = (\Delta u + f)_+$ at (x_0, y_0), which contradicts (90). Thus, we obtain $\partial_t u - \Delta u - f > 0$ in $Q_0 \cap Q^*$ by (92).

Since $Q_T = Q_0 \cup Q_1 \cup Q_2$ is a disjoint union and $\mathcal{H}^{n-1}(Q_T \setminus Q^*) = 0$, we have

$$0 \overset{\text{(V4)}}{=} \iint_{Q_T} (\partial_t u - \Delta u - f)\, \partial_t u \, \mathrm{d}x\, \mathrm{d}t = \iint_{Q_0} (\partial_t u - \Delta u - f)\, \partial_t u \, \mathrm{d}x\, \mathrm{d}t$$
$$= \iint_{Q_0 \cap Q^*} (\partial_t u - \Delta u - f)\, \partial_t u \, \mathrm{d}x\, \mathrm{d}t.$$

By (89), we obtain $\mathcal{H}^{n+1}(Q_0 \cap Q^*) = 0$; otherwise the last integral is positive. Hence we conclude that

$$\mathcal{H}^{n+1}(Q_0) = \mathcal{H}^{n+1}(Q_0 \cap Q^*) + \mathcal{H}^{n+1}(Q_0 \setminus Q^*) = 0.$$

This completes the proof. □

Finally, let us discuss a possible formulation of (6) in the sense of viscosity solutions. Set

$$F(x,t,Y) := -\big(\operatorname{tr} Y + f(x,t)\big)_+ \quad \text{for } x \in \Omega,\ t \in (0,T),\ Y \in \mathbb{R}^{n\times n}_{\mathrm{sym}}, \tag{93}$$

where $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ denotes the set of all symmetric $n \times n$ real matrices. Then (1) is also written as

$$\partial_t u(x,t) + F(x,t,D^2u(x,t)) = 0,$$

where $D^2u(x,t) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is the Hessian matrix of u. Since F is degenerate elliptic, one may apply the theory of viscosity solutions to prove the existence and uniqueness of viscosity solutions of (93) under suitable assumptions for $f(x,t)$ and the boundary condition. However, to the authors' knowledge, no result on such a viscosity approach to (6) has been obtained except for [58] (see also [45] for the case $f \equiv 0$). Moreover, the relation between the notion of viscosity solutions and that of strong solutions for (6) is widely open. For further details of the theory of viscosity solutions, we refer the reader to [24], [41], [31] and references therein.

Acknowledgments

The authors are grateful to the anonymous referees for careful reading and useful comments to improve the presentation of the paper. G.A. is supported by JSPS KAKENHI Grant Number JP16H03946, JP16K05199, JP17H01095, by the Alexander von Humboldt Foundation and by the Carl Friedrich von Siemens Foundation. M.K. is supported by JSPS KAKENHI Grant Number JP17H02857. Both authors are also supported by the JSPS–CNR bilateral joint research project: *Innovative Variational Methods for Evolution Equations*.

Appendix A. Some auxiliary facts

Let us start with the following:

Lemma A.1. *Let $1 \le p \le \infty$, $\mu \in \mathcal{M}(\overline{\Omega})$ and $\zeta \in L^p(\Omega)$. It holds that $\mu_+ \in L^p(\Omega)$ if and only if $(\mu+\zeta)_+ \in L^p(\Omega)$. Moreover, $\|(\mu+\zeta)_+\|_{L^p(\Omega)} \le \|\mu_+\|_{L^p(\Omega)} + \|\zeta_+\|_{L^p(\Omega)}$. An analogous conclusion also holds for the negative part.*

To prove this lemma, we claim that:

Lemma A.2. *Let $\mu \in \mathcal{M}(\overline{\Omega})$ and $1 \le p \le \infty$. If there exists $f \in L^p(\Omega)$ such that*

$$0 \le \mu(B) \le \int_B f\,\mathrm{d}x \quad \text{for any Borel set } B \subset \overline{\Omega},$$

then μ is absolutely continuous (with respect to Lebesgue measure) with an L^p density function.

Proof. For any Borel set $\omega \subset \overline{\Omega}$ whose Lebesgue measure is zero, we find $\mu(\omega) = 0$ by assumption. Hence μ is absolutely continuous (with respect to Lebesgue measure), and therefore, there exists $g_\mu \in L^1(\Omega)$ such that

$$0 \le \mu(B) = \int_B g_\mu \, \mathrm{d}x \le \int_B f \, \mathrm{d}x$$

for any Borel set $B \subset \overline{\Omega}$. Thus we obtain $0 \le g_\mu \le f$, which implies $g_\mu \in L^p(\Omega)$ by $f \in L^p(\Omega)$. $\square$

Proof for Lemma A.1. Let $\zeta \in L^p(\Omega)$. Assume that μ_+ has an $L^p(\Omega)$ density function g. Denote by $\mathcal{B}$ the set of all Borel sets in $\mathbb{R}^n$. We then observe that

$$\begin{aligned} 0 \le (\mu+\zeta)_+(B) &:= \sup_{\substack{B' \subset B \\ B' \in \mathcal{B}}} (\mu+\zeta)(B') \\ &= \sup_{\substack{B' \subset B \\ B' \in \mathcal{B}}} \left[\mu(B') + \int_{B'} \zeta \, \mathrm{d}x \right] \\ &\le \sup_{\substack{B' \subset B \\ B' \in \mathcal{B}}} \mu(B') + \int_B \zeta_+ \, \mathrm{d}x \\ &= \mu_+(B) + \int_B \zeta_+ \, \mathrm{d}x = \int_B (g + \zeta_+) \, \mathrm{d}x \end{aligned}$$

for any Borel set $B \subset \overline{\Omega}$. Therefore by Lemma A.2, $(\mu+\zeta)_+$ turns out to be absolutely continuous (with respect to Lebesgue measure) with an L^p density function h. Moreover, from the arbitrariness of B, one has $0 \le h \le g + \zeta_+$ a.e. in Ω, which also yields $\|h\|_{L^p(\Omega)} \le \|g\|_{L^p(\Omega)} + \|\zeta_+\|_{L^p(\Omega)}$. Conversely, assume that $(\mu+\zeta)_+ \in L^p(\Omega)$. Then

$$\begin{aligned} 0 \le \mu_+(B) &:= \sup_{\substack{B' \subset B \\ B' \in \mathcal{B}}} \mu(B') \\ &= \sup_{\substack{B' \subset B \\ B' \in \mathcal{B}}} \left[\mu(B') + \int_{B'} \zeta \, \mathrm{d}x - \int_{B'} \zeta \, \mathrm{d}x \right] \le (\mu+\zeta)_+(B) + \int_B \zeta_- \, \mathrm{d}x \end{aligned}$$

for any Borel set $B \subset \overline{\Omega}$. Thus μ_+ is absolutely continuous (with respect to Lebesgue measure) and has an L^p density function. $\square$

Finally, let us give a couple of concrete examples of ψ satisfying (24).

Remark A.3.

(i) If $\psi \in W^{2,1}(\Omega)$, $(-\Delta\psi)_+ \in L^p(\Omega)$ and $(\partial_\nu \psi)_+ = 0$ $\mathcal{H}^{n-1}$-a.e. on Γ_N, then (24) follows immediately.

(ii) Let us give another example of ψ satisfying (23) and (24): Let D be a smooth bounded domain of $\mathbb{R}^{n-1}$ and set $\Omega = D \times (-1, 1)$, $\Gamma_D = D \times \{-1\}$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Moreover, set

$$\psi(x) = |x_n| - 1 - a(x_n + 1) - b\left\{\frac{1}{4}(x_n - 1)^2 - 1\right\} \quad \text{for } x = (x', x_n) \in D \times (-1, 1)$$

for arbitrary $a \geq 1$ and $b \in \mathbb{R}$. Then for any $\varphi \in V$ (in particular, $\varphi(x', -1) \equiv 0$),

$$\langle A\psi - f, \varphi \rangle_V = -2 \int_D \varphi(x', 0)\, \mathrm{d}x' - (a-1) \int_D \varphi(x', 1)\, \mathrm{d}x' + \int_\Omega \left(\sigma\psi - f + \frac{b}{2}\right) \varphi\, \mathrm{d}x.$$

Hence, denote $A\psi - f$ by $\mu \in \mathcal{M}(\overline{\Omega})$. Then $\mathrm{d}\mu_+ = (\sigma\psi - f + \frac{b}{2})_+ \, \mathrm{d}x$, which is absolutely continuous (with respect to Lebesgue measure) and

$$\mathrm{d}\mu_- = 2\, \mathrm{d}\mathcal{H}^{n-1} \lfloor_{D\times\{0\}} + (a-1)\mathrm{d}\mathcal{H}^{n-1} \lfloor_{D\times\{1\}} + \left(\sigma\psi - f + \frac{b}{2}\right)_- \mathrm{d}x.$$

References

[1] S. Aizicovici, Q. Yan, Convergence theorems for abstract doubly nonlinear differential equations, Panamer. Math. J. 7 (1997) 1–17.
[2] G. Akagi, Doubly nonlinear evolution equations with non-monotone perturbations in reflexive Banach spaces, J. Evol. Equ. 11 (2011) 1–41.
[3] G. Akagi, Global attractors for doubly nonlinear evolution equations with non-monotone perturbations, J. Differential Equations 250 (2011) 1850–1875.
[4] G. Akagi, U. Stefanelli, Weighted energy-dissipation functionals for doubly nonlinear evolution, J. Funct. Anal. 260 (2011) 2541–2578.
[5] G. Akagi, U. Stefanelli, Periodic solutions for doubly nonlinear evolution equations, J. Differential Equations 251 (2011) 1790–1812.
[6] L. Ambrosio, V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, Comm. Pure Appl. Math. 43 (1990) 999–1036.
[7] L. Ambrosio, V.M. Tortorelli, On the approximation of free discontinuity problems, Boll. Unione Mat. Ital. 6-B (1992) 105–123.
[8] T. Arai, On the existence of the solution for $\partial\varphi(u'(t)) + \partial\psi(u(t)) \ni f(t)$, J. Fac. Sci. Univ. Tokyo Sec. IA Math. 26 (1979) 75–96.
[9] M. Aso, M. Frémond, N. Kenmochi, Phase change problems with temperature dependent constraints for the volume fraction velocities, Nonlinear Anal. 60 (2005) 1003–1023.
[10] M. Aso, N. Kenmochhi, Quasivariational evolution inequalities for a class of reaction–diffusion systems, Nonlinear Anal. 63 (2005) 1207–1217.
[11] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, 1976.
[12] V. Barbu, Existence theorems for a class of two point boundary problems, J. Differential Equations 17 (1975) 236–257.
[13] G.I. Barenblatt, V.M. Prostokishin, A mathematical model of damage accumulation taking into account microstructural effects, European J. Appl. Math. 4 (1993) 225–240.
[14] M. Bertsch, P. Bisegna, Blow-up of solutions of a nonlinear parabolic equation in damage mechanics, European J. Appl. Math. 8 (1997) 89–123.
[15] E. Bonetti, G. Schimperna, Local existence for Frémond's model of damage in elastic materials, Contin. Mech. Thermodyn. 16 (2004) 319–335.
[16] E. Bonetti, G. Schimperna, A. Segatti, On a doubly nonlinear model for the evolution of damaging in viscoelastic materials, J. Differential Equations 218 (2005) 91–116.
[17] G. Bonfanti, F. Luterotti, Well-posedness results and asymptotic behaviour for a phase transition model taking into account microscopic accelerations, J. Math. Anal. Appl. 320 (2006) 95–107.
[18] H. Brézis, Opérateurs maximaux monotones et sémi-groupes de contractions dans les éspaces de Hilbert, Math. Studies, vol. 5, North-Holland, Amsterdam/New York, 1973.
[19] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.

[20] J.-F. Babadjian, V. Millot, Unilateral gradient flow of the Ambrosio–Tortorelli functional by minimizing movements, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014) 779–822.
[21] H. Brézis, M.G. Crandall, A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure Appl. Math. 23 (1970) 123–144.
[22] P. Colli, On some doubly nonlinear evolution equations in Banach spaces, Jpn. J. Ind. Appl. Math. 9 (1992) 181–203.
[23] P. Colli, A. Visintin, On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations 15 (1990) 737–756.
[24] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[25] G.A. Francfort, J.-J. Marigo, Revisiting brittle fracture as an energy minimization problem, J. Mech. Phys. Solids 46 (1998) 1319–1342.
[26] M. Frémond, Non-smooth Thermomechanics, Springer-Verlag, Berlin, 2002.
[27] A. Friedman, Variational Principles and Free-Boundary Problems, A Wiley-Interscience Publication. Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1982.
[28] A. Giacomini, Ambrosio–Tortorelli approximation of quasi-static evolution of brittle fractures, Calc. Var. Partial Differential Equations 22 (2005) 129–172.
[29] U. Gianazza, G. Savaré, Some results on minimizing movements, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 112 (1994) 57–80.
[30] U. Gianazza, M. Gobbino, G. Savaré, Evolution problems and minimizing movements, Rend. Mat. Accad. Lincei IX 5 (1994) 289–296.
[31] Y. Giga, Surface Evolution Equations: A Level Set Approach, Monographs in Mathematics, vol. 99, Birkhäuser Verlag, Basel, 2006.
[32] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 edition Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[33] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[34] B. Gustafsson, A simple proof of the regularity theorem for the variational inequality of the obstacle problem, Nonlinear Anal. 10 (1986) 1487–1490.
[35] L.M. Kachanov, Introduction to Continuum Damage Mechanics, Mechanics of Elastic Stability, vol. 10, Springer, Netherlands, 1986.
[36] M. Kimura, T. Takaishi, Phase field models for crack propagation, Theor. Appl. Mec. Jpn. 59 (2011) 85–90.
[37] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Pure and Applied Mathematics, vol. 88, Academic Press, Inc., New York–London, 1980.
[38] D. Knees, A. Mielke, C. Zanini, On the inviscid limit of a model for crack propagation, Math. Models Methods Appl. Sci. 18 (2008) 1529–1569.
[39] D. Knees, R. Rossi, C. Zanini, A vanishing viscosity approach to a rate-independent damage model, Math. Models Methods Appl. Sci. 23 (2013) 565–616.
[40] D. Knees, R. Rossi, C. Zanini, A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains, Nonlinear Anal. Ser. B Real World Appl. 24 (2015) 126–162.
[41] S. Koike, A Beginner's Guide to the Theory of Viscosity Solutions, MSJ Memoirs, vol. 13, Mathematical Society of Japan, Tokyo, 2004.
[42] H. Lewy, G. Stampacchia, On the smoothness of superharmonics which solve a minimum problem, J. Anal. Math. 23 (1970) 227–236.
[43] M.W. Licht, Smoothed projections and mixed boundary conditions, arXiv:1710.06868.
[44] J.-L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Vol. I, Translated from the French by P. Kenneth Die Grundlehren der mathematischen Wissenschaften, vol. 181, Springer-Verlag, New York–Heidelberg, 1972.
[45] Q. Liu, Waiting time effect for motion by positive second derivatives and applications, NoDEA Nonlinear Differential Equations Appl. 21 (2014) 589–620.
[46] A. Mielke, A New Approach to Elasto-Plasticity Using Energy and Dissipation Functionals, Applied Mathematics Entering the 21st Century, SIAM, Philadelphia, PA, 2004, pp. 315–335.
[47] A. Mielke, R. Rossi, Existence and uniqueness results for a class of rate-independent hysteresis problems, Math. Models Methods Appl. Sci. 17 (2007) 81–123.
[48] A. Mielke Alexander, T. Roubíček, U. Stefanelli, Γ-limits and relaxations for rate-independent evolutionary problems, Calc. Var. Partial Differential Equations 31 (2008) 387–416.
[49] A. Mielke, F. Theil, On rate-independent hysteresis models, NoDEA Nonlinear Differential Equations Appl. 11 (2004) 151–189.

[50] E. Rocca, R. Rossi, Entropic solutions to a thermodynamically consistent PDE system for phase transitions and damage, SIAM J. Math. Anal. 47 (2015) 2519–2586.
[51] T. Roubíček, Nonlinear Partial Differential Equations with Applications, International Series of Numerical Mathematics, vol. 153, Birkhäuser Verlag, Basel, 2005.
[52] G. Schimperna, A. Segatti, U. Stefanelli, Well-posedness and long-time behavior for a class of doubly nonlinear equations, Discrete Contin. Dyn. Syst. 18 (2007) 15–38.
[53] A. Segatti, Global attractor for a class of doubly nonlinear abstract evolution equations, Discrete Contin. Dyn. Syst. 14 (2006) 801–820.
[54] T. Senba, On some nonlinear evolution equation, Funkcial. Ekvac. 29 (1986) 243–257.
[55] U. Stefanelli, The Brezis–Ekeland principle for doubly nonlinear equations, SIAM J. Control Optim. 47 (2008) 1615–1642.
[56] T. Takaishi, Numerical simulations of a phase field model for mode III crack growth (in Japanese), Trans. Jpn. Soc. Indust. Appl. Math. 19 (2009) 351–369.
[57] T. Takaishi, M. Kimura, Phase field model for mode III crack growth in two dimensional elasticity, Kybernetika 45 (2009) 605–614.
[58] N. Yamada, An Idea of the Proof of the Comparison Principle of Viscosity Solutions for Doubly Nonlinear Hamilton–Jacobi Equations, Proceedings of Seminar on Partial Differential Equations in Osaka 2012, Osaka University, 2013.