Technische Universität München
Fakultät für Mathematik
Lehrstuhl für Biomathematik

On Representation and Uniqueness of Invariant Means on Hypergroups

Michael Hofmann

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzende:		Univ.-Prof. Dr. Nina Gantert
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Rupert Lasser
	2.	Prof. Anthony To-Ming Lau, Ph.D., University of Alberta, Canada
	3.	Prof. Bertram M. Schreiber, Wayne State University, Detroit, USA

Die Dissertation wurde am 27. Oktober 2011 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 25. Mai 2012 angenommen.

Abstract

In this thesis we will transfer some of the known results from amenability on locally compact groups to polynomial hypergroups. We define summing sequences, an analogue to Følner sequences on groups, and slightly weaker conditions (F_p) and find criteria when these conditions are satisfied. With their help we prove two main representation theorems that allow to explicitly calculate mean values. We will then give some results about when all means coincide for certain functions, e.g. for weakly almost periodic functions, and prove a central result that on polynomial hypergroups there always exists more than one mean so that global uniqueness of means is not possible. We also introduce and investigate a notion we call 'strong amenability' that allow the translation operator to be shifted in the mean of a product of functions, inspired by some of the previous results about summing sequences that prove more than normal translation invariance.

Zusammenfassung

In dieser Dissertation übertragen wir einige bekannte Ergebnisse der Mittelbarkeit von lokalkompakten Abelschen Gruppen auf polynomiale Hypergruppen. Wir führen Summationsfolgen ein, analog den Følner-Folgen auf Gruppen, und etwas schwächere Bedingungen (F_p) und geben hinreichende und notwendige Bedingungen an, wann diese gelten. Mithilfe dieser Konstruktionen beweisen wir zwei wichtige Darstellungssätze, die explizites Ausrechnen von Mittelwerten ermöglichen. Weiter zeigen wir einige Resultate über die Eindeutigkeit von Mittelwerten, zum Beispiel für schwach fast periodische Funktionen, und beweisen ein zentrales Ergebnis, dass es auf polynomialen Hypergruppen immer mehr als einen Mittelwert gibt und somit globale Eindeutigkeit nicht gegeben ist. Angeleitet durch ein voriges Resultat, das mehr als normale Mittelbarkeit für Summationsfolgen gezeigt hat, definieren wir das Konzept der "starken Mittelbarkeit", die es erlaubt, den Translationsoperator in der Mittelwertbildung eines Produkts von Funktionen von einer Funktion zur anderen zu verschieben.

Acknowledgement

This dissertation would not have been possible without the help and support from many people to whom I am greatly indebted.

First and foremost, I am particularly grateful to my doctoral advisor, Professor Rupert Lasser, for his support, encouragement and patience throughout my doctoral studies. He provided a comfortable and productive working environment in his Institute for Biomathematics and Biometry at the HelmholtzZentrum münchen and was open for questions and advice whenever needed.

Most sincere thanks go to PD Dr. Volker Hösel without whom this dissertation would not have been possible. He patiently spent many hours discussing mathematical problems with me, answered questions and proposed valuable ideas.

Furthermore I want to thank PD Dr. Josef Obermaier for many enriching discussions and his continuing friendly support that improved my work substantially.

I am happy that I could share the time at Rupert Lasser's Institute of Biomathematics and Biometry with some great colleagues and I want to thank all of them for their great support, mathematical and personal. Especially I want to thank Nelson and Stefanie for the great time I had with fruitful discussions and your encouragement.

Many personal thanks go to my friends Annelie, Irmingard and Kristine for sharing the good times and keeping my morale up in bad times (not only) while writing this thesis and generally for just being there!

Finally I want to express my sincere gratitude to my family, Günter, Ursula and Andrea, for their love and invaluable support throughout my whole life. I am glad you are there!

Contents

Introduction

Averages and means appear in many different forms in mathematics. A typical question, for example coming from calculating means for stochastic processes, that arises for real-valued functions defined on the real line or on the set of integers would be: What is the average value of that function? For example, we would say a constant function $f \equiv c$ has average value c or a sine function has average 0 as it oscillates evenly around zero. We would like for such an average to have certain properties like monotonicity, i.e. if g dominates f then its average should also be greater, or translation invariance: If we shift a function (like the sine, to get e.g. the cosine) by a certain amount to the left or right, the average should stay the same.

The formal study of averages with these properties is the field of *amenability* or *amenable groups*. This English translation for the German word "messbar" (literally "measurable"), which has been introduced by John von Neumann in 1929 [30], has been given by Mahlon M. Day in 1949 in an abstract for an AMS meeting and is by many believed to be a pun: Amenable group is a group where you are *able* to find a *mean*, in contrast to the use as a synonym to manageable, complying.

Amenable groups are by definition those locally compact groups G that admit a (left) translation invariant monotone normalized (i.e. $m(\mathbf{1}) = 1$) linear functional m on $L^\infty(G)$. These are exactly the sensible requirements mentioned above.

One can say that the study of amenability began in 1904 when Henri Léon Lebesgue [23] introduced his integral on $\mathbb{R}$ and a set of conditions that characterize it. All but one of these are properties shared with the Riemann integral. The exceptional property is in essence the Monotone Convergence Theorem and Lebesgue asked whether it was possible to drop this condition without losing uniqueness of the Lebesgue integral. As the Monotone Convergence Theorem is equivalent to the countable additivity of the underlying Lebesgue measure from which the integral is constructed, the question is whether the Lebesgue integral is still unique if the Monotone Convergence Theorem condition is replaced by finite additivity. In 1923 Stefan Banach [2] answered this question negatively by constructing a finitely additive positive translation invariant measure μ on the family of bounded subsets of $\mathbb{R}$ such that $\mu([0, 1]) = 1$ but the corresponding integral does not coincide with the Lebesgue integral.

Naturally the question of the existence of invariant measures has been generalized to sets other than $\mathbb{R}$. In more general notation, take a group G acting on a set X and a subset $A \subseteq X$ used for normalizing the measure. Then the question is whether there exists an invariant measure on (G, X, A), i.e. a finitely additive measure $\mu : \mathcal{P}(X) \to [0, \infty]$ such that $\mu(xB) = \mu(B)$ for all $x \in G, B \in \mathcal{P}(X)$, and $\mu(A) = 1$. In 1914 Felix Hausdorff [12] worked on the problem of the existence of an invariant measure on $(G_n, \mathbb{R}^n, [0,1]^n)$ where G_n is the group of isometries (rather than translations, as invariance under isometries is more natural for the physical world) on $\mathbb{R}^n$. He was able to show that there exists no such measure for $n \geq 3$. The proof involves the construction of a partition $\{P, S_1, S_2, S_3\}$ of the 2-sphere and elements $\varphi, \psi \in SO(3)$ such that P is countable and

$$\varphi(S_1) = S_2 \cup S_3, \quad \psi(S_1) = S_2, \quad \psi^2(S_1) = S_3.$$

The positive answer for the cases $n = 1, 2$ was given by Banach in 1923 [2]. The crucial difference between these cases is that $SO(3)$ is a subgroup of G_n if and only if $n \geq 3$ and that makes the construction given by Hausdorff possible for all $n \geq 3$. This is also what is behind the famous paradox established by Banach and Alfred Tarski in 1924 [3] which in an exaggerated form states that a billiard ball can be cut into a finite number of pieces and putting them together in a different way produces a life-size statue of Banach. Or, mathematically speaking, the Banach-Tarski Theorem states that if $X, Y \subset \mathbb{R}^3$ are bounded with nonempty interior, then X and Y are congruent. The point is that the partitions involved in these operations contain nonmeasurable sets since the Lebesgue measure preserves volume. Tarski further explored the issue of paradoxical cuts and in 1938 [29] proved that there exists an invariant measure for (G, X, A) if and only A admits no paradoxical decomposition (with respect to G). A set A is said to admit a paradoxical decomposition if there exists a partition $\{A_1, \ldots, A_m, B_1, \ldots, B_n\}$ of A and elements $x_1, \ldots, x_m, y_1, \ldots, y_n \in G$ such that $\{x_1 A_1, \ldots, x_m A_m\}$ as well as $\{y_1 B_1 \ldots, y_n B_n\}$ are partitions of A. One can see why this is called "paradoxical" and why there cannot exist an invariant measure for (G, X, A). For assume μ were such a measure, then

$$\begin{aligned} 1 &= \mu(A) = \sum_{i=1}^{m} \mu(A_i) + \sum_{j=1}^{n} \mu(B_j) = \sum_{i=1}^{m} \mu(x_i A_i) + \sum_{j=1}^{n} \mu(y_j B_j) \\ &= \mu\left(\bigcup_{i=1}^{m} x_i A_i\right) + \mu\left(\bigcup_{j=1}^{n} y_j B_j\right) = 2\mu(A) = 2. \end{aligned}$$

The link between these early results about the existence of finitely additive measures and the existence of a mean is that a group is amenable if and only if there exists an invariant measure for (G, G, G). This change of perspective from finitely additive measures on G to functionals on $L^\infty(G)$ allows the use of many powerful tools of functional analysis to the study of amenability. To see how this transition works, let μ be a positive, finitely additive measure on $\mathcal{M}(G)$, the set of λ-measurable sets (where λ is a left Haar measure on G), that vanishes on locally null sets with $\mu(G) = 1$. Then μ can be regarded as an element m of $L^\infty(G)^*$ by first defining $m(\chi_A) := \mu(A)$ for $A \in \mathcal{M}(G)$. This can be immediately extended to the linear span of $\{\chi_A : A \in \mathcal{M}(G)\}$ and by continuity of m to all of $L^\infty(G)$, as the linear span of characteristic functions is norm dense in $L^\infty(G)$. The translation invariance of μ, i.e. $\mu(x^{-1}A) = \mu(A)$, leads to $m(\chi_A x) = m(\chi_A)$ as the translation of a characteristic function is again a characteristic function $\chi_A x = \chi_{x^{-1}A}$. Conversely, a translation invariant mean $m \in L^\infty(G)^*$ defines such a finitely additive positive measure μ by $\mu(A) := m(\chi_A)$ for $A \in \mathcal{M}(G)$.

One problem when switching to elements of $L^\infty(G)^*$ is that this space is very large and its structure is not very well understood. That is the reason one looks for a subset of $L^\infty(G)^*$ that is on the one hand comfortable enough to work with and on the other hand large enough to get all means from it. Utilizing $L^1(G)^* = L^\infty(G)$, isometrically embedding $L^1(G)$ in its second dual $L^\infty(G)^*$ by the canonical map $f \to \hat{f}, \hat{f}(\varphi) = \varphi(f)$ for $\varphi \in L^\infty(G)$, and looking for characterizations when functions $f \in L^1(G)$ give rise to means on $L^\infty(G)$ this leads to the definition of the set $P(G) = \{f \in L^1(G) : f \geq 0, \int f d\lambda = 1\}$. As $P(G)$ being a subset of $L^1(G)$ makes it an accessible set and on the other hand $\widehat{P}(G)$ is weak* dense in the set of means on $L^\infty(G)$ [24] this shows $P(G)$ to be a good choice in this context. All means on $L^\infty(G)$ are weak* limits of elements of $\widehat{P}(G)$.

For σ-compact groups one can get more tangible objects than weak* limits: Sequences of compact sets $(K_n)_{n\in\mathbb{N}_0}$ in G such that when one integrates a function over such a K_n then the result is arbitrarily close to being translation invariant. These sets are named 'Følner sequences' after Erling Følner [11]. They are ascending sequences of nonempty, compact sets such that $\lambda(xK_n \Delta K_n)/\lambda(K_n) \to 0$ for all $x \in G$, i.e. the differences between the shifted sets xK_n and K_n are small compared to the original sets K_n for $n \to \infty$. For σ-compact groups G amenability of a group is equivalent to the existence of a Følner sequence [24, Theorem (4.16)]. One classical example for that is $G = \mathbb{R}$ with the Følner sequence $K_n := [-n, n]$. For $\varphi \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$ one gets

$$\frac{1}{2n}\left|\int_{-n}^{n}(\varphi(t+x)-\varphi(t))dt\right| \leq \frac{2|x|\|\varphi\|_\infty}{2n} \to 0 \text{ as } n \to \infty$$

so integrating over $[-n, n]$, dividing by $\lambda([-n, n]) = 2n$ and taking the limit $n \to \infty$ yields a translation invariant mean on $\mathbb{R}$. This is the most direct way of getting a translation invariant mean on $\mathbb{R}$ as the only simpler way, directly integrating over $\mathbb{R}$ and then dividing by $\lambda(\mathbb{R}) = \infty$, is not possible.

One has to be careful that in general there exists not only one but many translation invariant means for a given group G. It seems difficult to give a general result, but a lower bound for the cardinality of the set of translation invariant means on noncompact amenable groups G is $2^{2^{\mathfrak{m}}}$ where $\mathfrak{m}$ is the smallest possible cardinality for a covering of G by compact subsets [24, Theorem (7.6)].

So in that case one can not hope for uniqueness of translation invariant means for the whole of $L^\infty(G)$ but only for a suitable subset. Recalling the properties we expect of a 'reasonable' definition of a mean in the beginning we clearly want constant functions to be in that subset. By the normalization $m(\mathbf{1}) = 1$ of all means this is satisfied. Another important class for which the means coincide is the set $\mathcal{WAP}(G)$ of weakly almost periodic functions on G [13, §18].

In contrast to periodic functions that repeat their values exactly in regular intervals almost periodic functions are in general only periodic up to a certain degree of accuracy so typical examples might be periodic functions with measurement errors or the sum of two periodic functions with noncommensurable periods. In 1925 Harald August Bohr [5] defined (uniformly) almost periodic functions on $\mathbb{R}$ as the closure of the trigonometric polynomials with respect to the supremum norm in $\mathcal{C}(\mathbb{R})$. He then proved the characterization given in words above, i.e. that a function f is almost periodic if and only if there exists a relative dense set of ε-almost periods for all $\varepsilon > 0$. An ε-almost period is a translation $T = T(\varepsilon)$ such that $|f(t+T) - f(t)| < \varepsilon$ for all t. There exist a number of additional characterizations of almost periodicity but the one most often used nowadays as definition on locally compact Abelian groups is that a function $f \in L^\infty(G)$ is almost periodic if the set of its translates $\{T_x f : x \in G\}$ forms a relative compact set in the norm topology. By taking the closure in the weak topology we get the larger class of weakly almost periodic functions $\mathcal{WAP}(G)$ on a locally compact Abelian group G. Weakly almost periodic functions are the largest set of elements in $L^\infty(G)$ for locally compact Abelian groups G for which we know that their mean is uniquely determined by the function alone (apart from the abstract definition of all functions whose mean is unique). For Abelian semigroups something similar holds (e.g. [7, 6]).

To study amenability on structures other than (semi)groups with a more general convolution replacing (semi)group operation a suitable structure is that of a hypergroup H where $x*y$ for $x, y \in H$ is a probability distribution over elements of the hypergroup rather than one element as in the group case. This (and also the other axioms of the definition of a hypergroup, see Chapter 1 for details) is satisfied by locally compact Abelian groups by identifying the group element $z := x \cdot y$ with its Dirac measure ε_z so hypergroups are indeed a generalization of groups. With hypergroups one can study various convolution structures on sets like $\mathbb{N}_0$ that do not allow for a group operation. For example as an index set for stochastic processes $\mathbb{N}_0$ is a more natural choice than $\mathbb{Z}$ (with its group structure), as usually measurements of time series start at some point t_0 and do not extend infinitely into the past. For this kind of processes some structural results have been found recently [14, 16]. For certain hypergroups on $\mathbb{N}_0$ there exists a relation between consecutive elements, the three-term recurrence relation. In that case, the hypergroup elements $n \in \mathbb{N}_0$ correspond to orthogonal polynomials R_n and for actual calculations involving convolutions on such a polynomial hypergroup one can utilize all knowledge about orthogonal polynomials [18, 19].

In this thesis we will transfer some of the known results from amenability on locally compact groups to polynomial hypergroups. We define summing sequences, an analogue to Følner sequences on groups, and the slightly weaker condition that a special sequence $(S_n)_{n\in\mathbb{N}_0}$ of sets satisfies one of the conditions named (F_p) and find criteria when these conditions are satisfied. With their help we prove two main representation theorems that allow to explicitly calculate mean values. We will then give some results about when all means coincide for certain functions, e.g. for weakly almost periodic functions, and prove a central result that on polynomial hypergroups there always exists more than one mean so that global uniqueness of means is not possible. We also introduce and investigate a notion we call ‘strong amenability’ that allow the translation operator to be shifted in a product of functions, i.e. $m(\varphi T_x \psi) = m(\psi T_{\tilde{x}} \varphi)$, inspired by some of the previous results about summing sequences that prove more than normal translation invariance.

In chapter 1 we present the definition of and some results on hypergroups we will need in later chapters. Special emphasis will be put on polynomial hypergroups and the link to the three-term recurrence relation of the underlying sequence of orthogonal polynomials. We then present the important subclass of polynomial hypergroups satisfying condition (H) and their relation to the growth of the Haar measure and conclude the chapter by presenting examples of polynomial hypergroups.

The second chapter starts with the definition of amenability on commutative hypergroups and results about the existence of means and uniqueness (for the class of weakly almost periodic functions) and studies the larger space of translation invariant functionals that are not necessarily means. The second section in this chapter presents the new condition of strong amenability that allows for the idea that $m(\varphi T_x \psi) = m(\psi T_{\tilde{x}} \varphi)$ as a replacement for a similar equation that cannot be transferred to hypergroups. Then we present a variation of Folner's condition P_1 for strong amenability and prove its equivalence to strong amenability.

Chapter 3 investigates amenability on polynomial hypergroups. First we present an alternate proof of the existence of translation invariant means that does not need the Markov-Kakutani fixed point theorem. We go on to see how the three-term recurrence relation facilitates proof of translation invariance and we show some sufficient conditions for the uniqueness of means. In the following section we present our concept of summing sequences, an analogue to Følner sequences on groups and introduce the canonical choice $(S_n)_{n\in\mathbb{N}_0}$ with $S_n := \{0, \dots, n\}$ and its relation to condition (H). Then we find positive as well as negative results about the existence of summing sequences and finally prove the main representation theorem 3.14. We conclude that section with a proof that hypergroups admitting summing sequences satisfy Reiter's condition (UFP_1). In section 3.2 we introduce conditions (F_p) on sequences of sets that are weaker than the assumption of existence of a summing sequence and prove results about existence of such sequences in a similar manner as in the previous section. We then give examples of polynomial hypergroups to illustrate the difference between conditions (F_p) and the existence of summing sequences. A representation theorem for condition (F_1) similar to the one for summing sequences also holds and we show that the existence of limits over summing sequences like $\lim\limits_{n\to\infty} \frac{1}{h(S_n)} \sum\limits_{k=0}^{n} \varphi(k)h(k)$ do not imply uniqueness of the mean for a given $\varphi \in l^\infty(\mathbb{N}_0)$. At the end of the chapter we show how difficult it is to improve the bound $\limsup\limits_{n\to\infty} \varphi_n \geq m(\varphi)$ for all means m by giving some counterexamples.

In chapter 4 we further explore the question whether it is possible to find a polynomial hypergroup where all means coincide. We introduce the concept of permanently positive sets for commutative hypergroups and with their help construct means that vanish for all functions whose translates lie in a certain ideal. We are able to find a single function χ_A independent of the specific choice of a polynomial hypergroup for which we can construct means M_1, M_2 such that $M_1(\chi_A) = 1$ and $M_2(\chi_A) = 0$. We conclude with the observation that if a summing sequence $(A_n)_{n\in\mathbb{N}_0}$ and the limit $\lim\limits_{n\to\infty} \frac{1}{h(A_n)} \sum\limits_{k\in A_n} \varphi(k)h(k)$ exist, then this limit is always the representation of a mean. For the hypergroup generated by the Chebyshev polynomials of the first kind we give two summing sequences that represent $M_1(\chi_A)$ and $M_2(\chi_A)$.

1 Basic facts about hypergroups

In this first chapter we will recall some basic facts and definitions about hypergroups for reference in later chapters. Classically, harmonic analysis is carried out on locally compact Abelian groups and a huge amount of theory has been developed for that case [13]. Locally compact Abelian groups are sets G that are both

- topological spaces satisfying the Hausdorff condition T_2 and such that every point in G has a neighborhood U such that its closure $\overline{U}$ is compact and
- groups with a commutative group operation such that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

But often one faces a situation where the underlying structure is a locally compact Hausdorff space and not a group but another algebraic structure is available which allows studying generalized translation operators, as it is e.g. the case with $\mathbb{N}_0$ which does not carry a group structure with the usual shift operator as there are no inverses. One such structure that preserves some properties of a translation operator is the notion of a hypergroup:

Definition 1.1. *Let K be a locally compact Hausdorff space and*

$$\omega : K \times K \to M^1(K)$$

a continuous mapping, where $M^1(K)$ carries the weak- topology induced by the relation $M^1(K) = C_0(K)^*$. For $\mu, \nu \in M(K)$ define*

$$\mu * \nu(f) := \int_{K\times K} \omega(x, y)(f) d(\mu \times \nu)(x, y)$$

for all $f \in C_0(K)$. Additionally, let

$$\tilde{\ } : K \to K, x \mapsto \tilde{x}$$

a homeomorphism and let $\tilde{\mu}(E) = \mu(\tilde{E})$ for $\mu \in M(K)$. Then the triple $(K, \omega, \tilde{\ })$ is called a **hypergroup** *if*

(H1) $\omega : K \times K \to M^1(K)$ *satisfies*

$$\varepsilon_x * \omega(y, z) = \omega(x, y) * \varepsilon_z$$

(H2) $\operatorname{supp}(\omega(x, y))$ *is compact for all $x, y \in K$*

(H3) $\tilde{\ } : K \to K$ *is an involution, i.e.* $\tilde{\tilde{x}} = x$ *and*

$$\widetilde{\omega(x, y)} = \omega(\tilde{y}, \tilde{x}) \qquad \forall x, y \in K$$

(H4) There exists a neutral element $e \in K$ such that

$$\omega(e,x) = \varepsilon_x = \omega(x,e) \qquad \forall x \in K$$

(H5) $e \in \operatorname{supp}(\omega(x,\tilde{y}))$ if and only if $x = y$

(H6) The mapping $(x,y) \mapsto \operatorname{supp}(\omega(x,y)), K \times K \to \mathcal{C}(K)$ is continuous

A hypergroup is called commutative if

$$\omega(x,y) = \omega(y,x)$$

for all $x, y \in K$.

Hypergroups are indeed a generalization of locally compact Abelian groups:

Remark 1.2. *A locally compact Abelian group G can be considered a commutative hypergroup by defining $\omega(x,y) := \varepsilon_{x \cdot y}$ and taking the inverse as involution $\tilde{x} := x^{-1}$.*

Similar to groups, on commutative hypergroups K there is a way to assign a translation invariant volume to Borel sets, the **Haar measure**:

Proposition 1.3. *If K is commutative there exists a (up to a multiplicative constant) unique regular positive Borel measure μ on K which is translation invariant, i.e.*

$$\int_K f(x) d\mu(x) = \int_K \omega(y,x) f(x) d\mu(x)$$

for all $y \in K$ and $f \in C_c(K)$.

For the proof, refer to [4, Theorems 1.3.15, 1.3.22].

Definition 1.4. *For $f, g \in L^1(K)$ we define a convolution by*

$$f * g(x) := \int_K f(y) \omega(\tilde{y}, x)(g) d\mu(y)$$

For easier notation we will also use the following form of a 'convolution of sets':

$$A * B := \bigcup_{a \in A, b \in B} \operatorname{supp}(\omega(a,b)) \text{ for } A, B \in K$$

It is important to note that we do not in any way take into account the relative weights that $\omega(a,b)$ distributes but just take its support. This notation is mostly used in proofs in which we want to get upper bounds for Haar measures of translates of sets (see Chapter 3).

1.1 Polynomial Hypergroups

We will mainly be concerned with polynomial hypergroups, i.e. hypergroups derived from sequences of orthogonal polynomials. Let $(R_n)_{n\in\mathbb{N}_0}$ be a sequence of polynomials that are orthogonal with respect to a probability measure $\pi \in M^1(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} R_n(x)R_m(x)d\pi(x) = \mu_n\delta_{m,n}$$

with $\mu_n > 0$. Often orthogonal polynomials are not defined by an orthogonalization measure but by a three-term recurrence relation

$$R_n(x)R_1(x) = a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x)$$

for $n \in \mathbb{N}$, and $R_0(x) = \mathbf{1}$, $R_1(x) = \frac{1}{a_0}(x - b_0)$, where $a_0 \neq 0$, $b_n \in \mathbb{R}$, $a_nc_{n+1} > 0$ for all $n \in \mathbb{N}_0$. Favard's Theorem assures the existence of a probability measure $\pi \in M^1(\mathbb{R})$ as above such that $(R_n)_{n\in\mathbb{N}_0}$ are orthogonal with respect to π.

Products of orthogonal polynomials can be linearized, as can be seen with the following Lemma (cf. eg. [10]).

Lemma 1.5. *Let $(R_n)_{n\in\mathbb{N}_0}$ be a sequence of polynomials that are orthogonal with respect to a probability measure $\pi \in M^1(\mathbb{R})$. Additionally, let $\deg(R_n) = n$ and $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. Then products $R_m(x)R_n(x)$ can be linearized by*

$$R_m(x)R_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n; k)R_k(x)$$

where $g(m, n; k) \in \mathbb{R}$ for $k = |m - n|, \ldots, m + n$. Additionally $g(m, n; |m - n|) \neq 0$ and $g(m, n; m + n) \neq 0$.

Proof. Obviously $(R_n)_{n\in\mathbb{N}_0}$ is an orthogonal basis of the space of all polynomials. Therefore products have a unique representation

$$R_m(x)R_n(x) = \sum_{k=0}^{\infty} g(m, n; k)R_k(x).$$

The assumption on the degree of the polynomials leads to $g(m, n; k) = 0$ for $k > m+n$ and $g(m, n; m + n) \neq 0$. Without loss of generality let $m > n$ and $k < m - n$. Then $\deg(R_nR_k) < m$ and thus

$$0 = \int_{\mathbb{R}} R_m(x)R_k(x)R_n(x)d\pi(x) = \sum_{j=0}^{m+n} g(m, n; j) \int_{\mathbb{R}} R_k(x)R_j(x)d\pi(x) = g(m, n; k)\mu_k$$

Since $\mu_k \neq 0$ for all $k \in \mathbb{N}_0$ we get $g(m, n; k) = 0$ for $k < |m - n|$. Now assume $g(m, n; m - n) = 0$. Then we get

$$\begin{aligned} 0 = \int_{\mathbb{R}} R_n(x)R_{m-n}(x)R_m(x)d\pi(x) &= \sum_{j=|m-2n|}^{m} g(n, m - n; j) \int_{\mathbb{R}} R_j(x)R_m(x)d\pi(x) \\ &= g(n, m - n; m)\mu_m \end{aligned}$$

which is a contradiction to $g(n, m - n; m) \neq 0$ and $\mu_m \neq 0$. □

The linearization coefficients satisfy some identities that will be of use later on, presented in the following Lemma.

Lemma 1.6. *Let* $(R_n)_{n\in\mathbb{N}_0}$ *and* $g(m,n;k)$ *as in Lemma 1.5. Then we get for* $k \in \{|m-n|,\dots,m+n\}$*:*

(i) $g(m,n;k) = g(n,m;k)$ *for all* $m,n \in \mathbb{N}_0$

(ii) $g(0,n;n) = 1$ *for all* $n \in \mathbb{N}_0$

(iii) $\sum_{j=|m-n|}^{m+n} g(m,n;j) = 1$ *for all* $m,n \in \mathbb{N}_0$

(iv) $g(m,n;k)\mu_k = g(m,k;n)\mu_n$ *for all* $m,n \in \mathbb{N}_0$

(v) $g(n,n;0) = \mu_n$ *for all* $n \in \mathbb{N}_0$

Proof. (i) Holds obviously as $R_m(x)R_n(x) = R_n(x)R_m(x)$

(ii) Since $R_0(x) = 1$ for all x, we get $R_0(x)R_n(x) = R_n(x)$

(iii) For all $m,n \in \mathbb{N}_0$ we get

$$1 = R_m(1)R_n(1) = \sum_{j=|m-n|}^{m+n} g(m,n;j)R_j(1) = \sum_{j=|m-n|}^{m+n} g(m,n;j)$$

(iv) Holds since

$$\begin{aligned}\int_{\mathbb{R}} R_m(x)R_n(x)R_k(x)d\pi(x) &= \sum_{j=|m-n|}^{m+n} g(m,n;j)\int_{\mathbb{R}} R_k(x)R_j(x)d\pi(x)\\ &= g(m,n;k)\mu_k\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} R_m(x)R_n(x)R_k(x)d\pi(x) &= \sum_{j=|m-k|}^{m+k} g(m,k;j)\int_{\mathbb{R}} R_n(x)R_j(x)d\pi(x)\\ &= g(m,k;n)\mu_n\end{aligned}$$

(v) Follows from (iv) and $\mu_0 = 1$

□

Orthogonal polynomials with a positivity condition on their linearization coefficients give rise to a hypergroup structure, as the following Theorem shows.

Theorem 1.7. *Let* $(R_n)_{n\in\mathbb{N}_0}$ *and* $g(m,n;k)$ *as in Lemma 1.5. Additionally, let*

$$g(m,n;k) \geq 0 \textit{ for } k = |m-n|,\dots,m+n.$$

Defining a convolution on $\mathbb{N}_0$ *(with the discrete topology) by*

$$\omega(m,n) := \sum_{k=|m-n|}^{m+n} g(m,n;k)\varepsilon_k \text{ for all } m,n \in \mathbb{N}_0$$

and the identity mapping as involution $\tilde{}$, *the triple* $(\mathbb{N}_0, \omega, \tilde{})$ *is a commutative hypergroup with neutral element* 0.

Proof. ω is a probability measure on $\mathbb{N}_0$ with compact support. By definition we have $0 \in \operatorname{supp}(\omega(m,n))$ if and only if $m = n$. Most of the other properties of Definition 1.1 are obviously satsified, only associativity (H1) remains to be shown. Therefore let $\mathbf{e}_j^k := \delta_{k,j}$. Then $\omega(m,n)(\mathbf{e}^k) = g(m,n;k)$ and we get

$$\begin{aligned}
(\varepsilon_l * \omega(m,n))(\mathbf{e}^k) &= \int_{\mathbb{N}_0}\int_{\mathbb{N}_0} \omega(x,y)(\mathbf{e}^k) d\varepsilon_l(x) d\omega(m,n)(y) = \int_{\mathbb{N}_0} \omega(l,y)(\mathbf{e}^k) d\omega(m,n)(y) \\
&= \sum_{j=0}^{\infty} \omega(l,j)(\mathbf{e}^k) g(m,n;j) = \sum_{j=0}^{\infty} g(l,j;k) g(m,n;j) \\
&= \frac{1}{\mu_k} \sum_{j=|m-n|}^{m+n} g(m,n;j) \int_{\mathbb{R}} R_l(x) R_j(x) R_k(x) d\pi(x) \\
&= \frac{1}{\mu_k} \int_{\mathbb{R}} R_l(x)(R_m(x)R_n(x)) R_k(x) d\pi(x) \\
&= \frac{1}{\mu_k} \int_{\mathbb{R}} (R_l(x)R_m(x)) R_n(x) R_k(x) d\pi(x) \\
&= \frac{1}{\mu_k} \sum_{j=|l-m|}^{l+m} g(l,m;j) \int_{\mathbb{R}} R_j(x) R_n(x) R_k(x) d\pi(x) \\
&= \sum_{j=|l-m|}^{l+m} g(l,m;j) g(j,n;k) = (\omega(l,m) * \varepsilon_n)(\mathbf{e}^k)
\end{aligned}$$

□

To check whether the additional assumption of nonnegative linearization coefficients is satisfied for a specific sequence of orthogonal polynomials one can for example find restrictions on the coefficients a_n, b_n, c_n of the three-term recurrence relation. In 1970 Askey [1] found such restrictions based on versions for monic polynomials and many generalizations of that result have since been given, for example by Szwarc [27, 28]. We will state one of the results that allows us to see that the ultraspherical polynomials generate polynomial hypergroups:

Theorem 1.8. *Let* $(R_n)_{n\in\mathbb{N}_0}$ *be given by*

$$R_0(x) = 1, R_1(x) = \frac{1}{a_0}(x - b_0),$$

$$R_n(x)R_1(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x) \text{ for } n \in \mathbb{N}$$

where $b_n \in \mathbb{R}$ *and* $a_n, c_n > 0$ *for all* $n \in \mathbb{N}_0$. *If in addition* *(i)* $c_n, b_n, a_n + c_n$ *are nondecreasing sequences*

(ii) $c_n \leq a_n$ *for all* $n \in \mathbb{N}$

then the linearization coefficients $g(n, m; k)$ *are nonnegative.*

Proof. This is a slight reformulation of [27, Theorem 1]. □

Example 1.9. *The coefficients of the three-term recurrence relation of the family of ultraspherical polynomials* $P_n^{(\alpha,\alpha)}$ *are given by*

$$\begin{aligned} a_0 &= 1, \quad b_0 = 0, \\ a_n &= \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad b_n = 0, \quad c_n = \frac{n}{2n+2\alpha+1} \text{ for } n \in \mathbb{N}. \end{aligned}$$

Since $\alpha \geq -\frac{1}{2}$ *and so* $2\alpha + 1 \geq 0$ *we get that* $c_n \leq a_n$ *and* c_n *is nondecreasing. And since* b_n *and* $a_n + c_n$ *are constant the ultraspherical polynomials satisfy (i) and (ii) of Theorem 1.8 and so they generate a polynomial hypergroup. More details on the ultraspherical polynomials can be found in Section 1.2.*

With the structure of a polynomial hypergroup, $\mathbb{N}_0$ admits a Haar measure h, i.e. a (up to a multiplicative constant) unique countably additive translation invariant measure such that $h(A)$ is finite for every finite A. The following theorem shows how the one with normalization $h(\{0\}) = 1$ can be obtained from knowledge about the underlying orthogonal polynomials.

Theorem 1.10. *Let* $\mathbb{N}_0$ *carry a polynomial hypergroup structure. Then the Haar measure is given by*

$$h(E) = \sum_{n \in E} \frac{1}{\mu_n} \text{ for all } E \subseteq \mathbb{N}_0$$

Proof. Using associativity, we get for $k, m, n \in \mathbb{N}_0$

$$\begin{aligned} \sum_{i=0}^{\infty} \omega(m,i)(\{0\})\,\omega(n,k)(\{i\}) &= (\varepsilon_m * \omega(n,k))(\{0\}) \\ &= (\omega(m,n) * \varepsilon_k)(\{0\}) \\ &= \sum_{i=0}^{\infty} \omega(m,n)(\{i\})\,\omega(i,k)(\{0\}) \end{aligned}$$

Since $0 \in \operatorname{supp}(m, n)$ if and only if $m = n$, the sums collapse to

$$\omega(m,m)(\{0\})\,\omega(n,k)(\{m\}) = \omega(m,n)(\{k\})\,\omega(k,k)(\{0\})$$

For convenience we set $h_1(m) := \omega(m,m)(\{0\})^{-1}$ and with Lemma 1.6 we see

$$h_1(m) = \omega(m,m)(\{0\})^{-1} = g(m,m;0)^{-1} = \frac{1}{\mu_m}$$

Finally, for $E \subseteq \mathbb{N}_0$ we define h by

$$h(E) := \sum_{m \in E} h_1(m) = \sum_{m \in E} \frac{1}{\mu_m}$$

Obviously $H : \mathcal{P}(\mathbb{N}_0) \to [0; \infty[$ is a positive measure on $\mathbb{N}_0$ and the equation above reads

$$h_1(k)\omega(n,k)(\{m\}) = h_1(m)\omega(m,n)(\{k\})$$

Translation invariance of h remains to be shown. It is sufficient to show

$$\int_{\mathbb{N}_0} \omega(n,k)(\{m\})dh(k) = h(\{m\})$$

for all $n \in \mathbb{N}_0$. This can be seen by

$$\begin{aligned}
\int_{\mathbb{N}_0} \omega(n,k)(\{m\})dh(k) &= \sum_{k=0}^{\infty} \omega(n,k)(\{m\})h_1(k) \\
&= \sum_{k=0}^{\infty} h_1(m)\omega(m,n)(\{k\}) = h_1(m) \sum_{k=0}^{\infty} \omega(m,n)(\{k\}) \\
&= h_1(m) \sum_{k=0}^{\infty} \sum_{j=|m-n|}^{m+n} g(m,n;j)\, \varepsilon_j(\{k\}) \\
&= h_1(m) \sum_{k=0}^{\infty} g(m,n;k) = h_1(m) = h(\{m\})
\end{aligned}$$

□

For convenience we will from now on write $h(n)$ instead of $h(\{n\})$ for one-point sets.

For sequences $(\beta_n)_{n \in \mathbb{N}_0}$ we define the translate $T_m\beta$ by

$$T_m\beta(n) := \sum_{k=|n-m|}^{n+m} g(n,m;k)\beta(k)$$

and the convolution of sequences $f, g \in l^1(h)$ is given by

$$f * g(n) = \sum_{k=0}^{\infty} T_n f(k) g(k) h(k)$$

The ‘convolution of sets’ used in some proofs looks like this:

$$A * B := \bigcup_{n \in A, m \in B} \operatorname{supp}(\varepsilon_n * \varepsilon_m)$$

For the special case of translates of sets we get

$$T_k A := \{k\} * A = \bigcup_{n \in A} \operatorname{supp}(\varepsilon_k * \varepsilon_n).$$

Thus it is clear that taking its Haar measure reads

$$h(T_k A) = h(\{k\} * A) = h\left(\bigcup_{n \in A} \operatorname{supp}(\varepsilon_k * \varepsilon_n)\right)$$

and no $g(k, n; .)$ appears representing the weights the convolution distributes. The reason one has to keep that in mind is that in contrast to the group case the translate $T_k \chi_A$ of a characteristic function is in general not anymore a characteristic function, in particular not equal to $\chi_{T_k A}$ for $k \neq 0$.

Example 1.11. *Let the hypergroup structure on $\mathbb{N}_0$ be given by the Chebyshev polynomials of the first kind $(T_n)_{n \in \mathbb{N}_0}$, i.e. $g(m, n; m+n) = g(m, n; |m-n|) = \frac{1}{2}$ for $m, n \neq 0$, and let $k \leq n$. Then we get*

$$T_k \chi_{\{0,\dots,n\}}(m) = \begin{cases} 1 & m = 0, \dots, n-k \\ \frac{1}{2} & m = n-k+1, \dots, m+n \\ 0 & m > n+k \end{cases}$$

whereas

$$\chi_{T_k\{0,\dots,n\}}(m) = \chi_{\{0,\dots,n+k\}} = \begin{cases} 1 & m = 0, \dots, n+k \\ 0 & m > n+k \end{cases} .$$

We do know, however, that $T_k \chi_A \leq \chi_{T_k A}$ for all $k \in \mathbb{N}_0, A \subseteq \mathbb{N}_0$ as whenever the left-hand side is not zero, then it must be between 0 and 1 as a convex combination of these numbers and the right-hand side must be 1.

An important property many polynomial hypergroups fulfil is a condition on the growth of the Haar measure. We call it property (H).

Definition 1.12. *The polynomial hypergroup $\mathbb{N}_0$ is said to satisfy property (H) if*

$$\lim_{n \to \infty} \frac{h(n)}{\sum_{k=0}^{n} h(k)} = 0$$

It is straightforward to see that property (H) is satisfied whenever $h(n)$ is growing polynomially, i.e. $h(n) = O(n^\alpha), \alpha \geq 0$ as $n \to \infty$, and that it is not satisfied if $h(n)$ grows exponentially.

If property (H) holds, similar conditions hold if the numerator in the expression for property (H) is replaced by $h(n+s)$ with $s \in \mathbb{N}_0$:

Proposition 1.13. *Assume that (H) is satisfied. Then*

$$\lim_{n \to \infty} \frac{h(n+s)}{\sum_{k=0}^{n} h(k)} = 0 \text{ for all } s \in \mathbb{N}_0.$$

Proof. First we will show that

$$H_s(n) := \frac{1}{h(n+s)} \sum_{k=0}^{n} h(k) \text{ for } s, n \in \mathbb{N}_0$$

satisfy the following recurrence formula:

$$H_s(n)(1 + \frac{1}{H_1(n)}) = H_{s-1}(n+1)$$

Note the identities

$$H_s(n) = H_{s-1}(n+1) - \frac{h(n+1)}{h(n+s)} \text{ and}$$

$$H_s(n) = H_{s-1}(n)\frac{h(n+s-1)}{h(n+s)} = \ldots = H_1(n)\frac{h(n+1)}{h(n+s)} = H_0(n)\frac{h(n)}{h(n+s)}.$$

By these two expressions follows

$$H_s(n) = H_{s-1}(n+1) - \frac{h(n+1)}{h(n+s)} = H_{s-1}(n+1) - \frac{H_s(n)}{H_1(n)}$$

and hence $H_s(n)(1 + \frac{1}{H_1(n)}) = H_{s-1}(n+1)$.

Now assume (H). We apply induction on s. For $s = 1$ we have

$$\frac{h(n+1)}{\sum_{k=0}^{n+1} h(k) - h(n+1)} = \frac{1}{\frac{\sum_{k=0}^{n+1} h(k)}{h(n+1)} - 1} \to 0 \text{ as } n \to \infty.$$

Suppose the statement is proven for $l = 0, \ldots, s-1$. Then by the recurrence formula from the beginning of the proof immediately follows $H_s(n) \to \infty$ as $n \to \infty$. □

As the growth condition (H) also implies a similar condition with $h(n+s)$ in the numerator for negative s, all conditions of the form $\frac{h(n+s)}{\sum_{k=0}^{n} h(k)} \to 0$ as $n \to \infty$ are equivalent.

1.2 Examples of Polynomial Hypergroups

In this section we will look at specific polynomial hypergroups and shortly examine their properties. The first and most common examples are hypergroups generated by Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $\alpha \geq \beta > -1$ and $\alpha+\beta+1 \geq 0$. Jacobi polynomials are orthogonal with respect to a measure $d\pi$ concentrated on $[-1, 1]$. This measure is given by

$$d\pi(x) = C_{(\alpha,\beta)}(1-x)^\alpha(1+x)^\beta \chi_{[-1,1]} dx$$

where $C_{(\alpha,\beta)} = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}$. Their coefficients in the three-term recurrence relation are

$$\begin{aligned}
a_0 &= \frac{2(\alpha+1)}{\alpha+\beta+2}, \quad b_0 = \frac{\beta-\alpha}{\alpha+\beta+1}, \\
a_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(\alpha+1)}, \\
b_n &= \frac{\alpha-\beta}{2(\alpha+1)} \left(1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right), \\
c_n &= \frac{n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)(\alpha+1)}.
\end{aligned}$$

Haar weights are given by

$$h(0) = 1, \quad h(n) = \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n(\alpha+1)_n}{(\alpha+\beta+1)n!(\beta+1)_n} \text{ for } n \geq 1.$$

An important special class of Jacobi polynomials are **ultraspherical polynomials** where $\alpha = \beta \geq -\frac{1}{2}$. The formulas here are valid for $\alpha = \beta \neq \frac{1}{2}$ and the ones for $\alpha = \beta = \frac{1}{2}$ can be found below. They are orthogonal with respect to the measure

$$d\pi(x) = C_\alpha (1-x^2)^\alpha \chi_{[-1,1]} dx$$

where $C_\alpha = \frac{1}{2^{2\alpha+1}} \frac{\Gamma(2\alpha+2)}{(\Gamma(\alpha+1))^2}$.
Their recursion coefficients are given by

$$\begin{aligned}
a_0 &= 1, \quad b_0 = 0, \\
a_n &= \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad b_n = 0, \quad c_n = \frac{n}{2n+2\alpha+1}.
\end{aligned}$$

Their linearization coefficients have the form (where $m \leq n$)

$$g(m,n;k) = \frac{n!m!(\alpha+\frac{1}{2})_k(\alpha+\frac{1}{2})_{n-k}(\alpha+\frac{1}{2})_{m-k}(2\alpha+1)_{n+m-k}(n+m+\alpha+\frac{1}{2}-2k)}{k!(n-k)!(m-k)!(\alpha+\frac{1}{2})_{n+m-k}(2\alpha+1)_n(2\alpha+1)_m(n+m+\alpha+\frac{1}{2}-k)}$$

if $k \in \{n-m, n-m+2, n-m+4, \ldots, n+m\}$ and

$$g(n,m;k) = 0$$

if $k \in \{n-m+1, n-m+3, n-m+5, \ldots, n+m-1\}$.
The Haar weights are

$$h(0) = 1, \quad h(n) = \frac{(2n+2\alpha+1)(2\alpha+1)_n}{(2\alpha+1)n!}$$

One very simple example of Jacobi polynomials are the **Chebyshev polynomials of the first kind**, where $\alpha = \beta = -\frac{1}{2}$. They satisfy

$$T_n(x) = \cos(n \arccos x).$$

They are orthogonal with respect to

$$d\pi(x) = \frac{dx}{\sqrt{1-x^2}}$$

Via trigonometric addition formulas one gets

$$T_m(x)T_n(x) = \frac{1}{2}T_{|m-n||}(x)T_{m+n}(x)$$

and so the linearization coefficients are given by

$$g(m,n;k) = \begin{cases} 1 & \text{if } mn = 0 \text{ and } k = \max(m,n) \\ \frac{1}{2} & \text{if } m,n \in \mathbb{N}_0 \text{ and } k = |m-n|, m+n \\ 0 & \text{else} \end{cases}$$

Haar weights are

$$h(0) = 1, h(n) = 2 \text{ for } n \in \mathbb{N}.$$

For $\alpha = \beta = \frac{1}{2}$ we get the **Chebyshev polynomials of the second kind**. They satisfy

$$U_n(x) = \frac{1}{n+1}\frac{\sin((n+1)\arccos x)}{\sin\arccos x}.$$

Chebyshev polynomials of the second kind are orthogonal with respect to

$$d\pi(x) = \sqrt{1-x^2}dx$$

and their linearization coefficients are given by

$$g(m,n;m+n-k) = \begin{cases} \frac{m+n+1-k}{(m+1)(n+1)} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

The Haar measure is

$$h(n) = (n+1)^2.$$

Another special case of Jacobi polynomials are **Legendre polynomials**, where $\alpha = \beta = 0$. They are given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

and they are orthogonal with respect to the Lebesgue measure on $[-1,1]$. The linearization coefficients are given by $g(m,n;m+n-j) = 0$ for $j \in \{1,3,\ldots,2\min(m,n)-1\}$ and

$$g(m,n;m+n-2j) = \frac{(m+n+\frac{1}{2}-2j)\binom{2j}{j}\binom{2m-2j}{m-j}\binom{2n-2j}{n-j}}{(m+n+\frac{1}{2}-j)\binom{2m+2n-2j}{m+n-j}} \text{ for } j \in \{0,1,\ldots,\min(m,n)\}.$$

The Haar measure is

$$h(n) = 2n+1.$$

A further interesting polynomial hypergroup with exponentially increasing Haar measure is generated by **Little q-Legendre polynomials** $R_n^{(q)}(x)$ with $0 < q < 1$. That they define a polynomial hypergroup on $\mathbb{N}_0$ is shown in [17] and [10]. The recurrence coefficients are

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}$$

$$b_n = \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})}$$

$$c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}$$

for $n \in \mathbb{N}$, with starting values $a_0 = \frac{1}{q+1}$ and $b_0 = \frac{q}{q+1}$. The Haar weights satisfy

$$\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = \frac{1}{q} > 1.$$

Hence $h(n)$ is of exponential growth. Moreover, $a_n \to 0, c_n \to 0, b_n \to 1$ and $\frac{a_n}{c_{n+1}} \to \frac{1}{q}$.

2 Translation invariant means on commutative hypergroups

2.1 Definitions and Basic Properties

The definition of a translation invariant mean here is the same as in the group case with the hypergroup convolution replacing the group action. We will still call this a translation.

Definition 2.1. *Let K be a commutative hypergroup. A linear functional m on $L^\infty(K)$ is called a* **mean** *if*

(i) $m(\bar{f}) = \overline{m(f)}$ *for all* $f \in L^\infty(K)$

(ii) $f \geq 0$ *implies* $m(f) \geq 0$

(iii) $m(\mathbf{1}) = 1$

The set of all means on K will be called $\mathfrak{M}(K)$ (or abbreviated just $\mathfrak{M}$). A mean is called **(left) translation invariant** *if $m(T_x\varphi) = m(\varphi)$ for all $x \in K, \varphi \in L^\infty(K)$. If such a translation invariant mean exists, K is called* **amenable**. *We will call $\mathfrak{M}_t(K)$ (or just $\mathfrak{M}_t$) the set of all (left and thus two-sided) translation invariant means on K.*

A linear functional $m \in (L^\infty(K))^*$ is a mean if and only if $m(\mathbf{1}) = 1 = \|m\|$ and so $\mathfrak{M}$ is a non-empty weak* compact convex set in $(L^\infty(K))^*$.
Commutative hypergroups are amenable, as there exists a (left) translation invariant mean. This can be shown by application of the Markov-Kakutani fixed point theorem, see e.g. [22, p. 168]. Another proof of this well-known fact, which uses the existence of a mean for the space of bounded continuous functions on any Abelian semigroup and embeds K into $P_{00}(K)$, the probability measures on K with compact support, can be found in [7].

Remark 2.2. *Let m be a translation invariant mean and $\varphi \in L^\infty(K)$ real-valued. Then the mean is bounded by $\|\varphi\|_\infty$:*

$$-\|\varphi\|_\infty \leq m(\varphi) \leq \|\varphi\|_\infty$$

This follows from the monotonicity of m.

For a certain class of functions $f \in L^\infty(K)$ the value $m(f)$ is uniquely defined for all means m:

Proposition 2.3. *Let $f \in L^\infty(K)$ such that a constant function $\tilde{f} \equiv c_f$ lies in $\overline{co\{T_x f : x \in K\}}^\infty$. Then $m(f) = c_f$ for all $m \in \mathfrak{M}_t$, i.e. $m(f)$ is uniquely determined and equal to this constant.*

Proof. Let m be a translation invariant mean on $L^\infty(K)$. For any convex combination of elements of $\{T_x f : x \in K\}$ the mean is the same and equal to $m(f)$. Since $m \in (L^\infty(K))^*$, this also holds for the norm closure of this set. If there is a constant function in this set, then $m(f)$ must be equal to this constant as m satisfies $m(\mathbf{1}) = 1$, regardless of the choice of m. □

With this Proposition in mind, we define the following sets of functions:

Definition 2.4. *Let K be a commutative hypergroup. Then define*

$$\begin{aligned} \mathcal{CO}(K) &:= \{f \in L^\infty(K) : \text{There is a constant function in } \overline{co\{T_x f : x \in K\}}^\infty\} \\ \mathcal{AC}(K) &:= \{f \in L^\infty(K) : \exists c_f \in \mathbb{C} : m(f) = c_f \text{ for all } m \in \mathfrak{M}_t\} \end{aligned}$$

The set $\mathcal{CO}(K)$ contains all functions that have a constant in the closure of the convex hull of their translates which is exactly the condition used in Proposition 2.3 to prove that all means coincide for these functions. $\mathcal{AC}(K)$ is the set of all functions $f \in L^\infty(K)$ for which all means coincide. We call elements in $\mathcal{AC}(K)$ almost convergent. This comes from the fact that all convergent sequences on polynomial hypergroups lie in $\mathcal{AC}(\mathbb{N}_0)$ which will be shown in Proposition 3.7 and that in this sense the elements of the larger set $\mathcal{AC}(\mathbb{N}_0)$ are similar, hence "almost" convergent.

So Proposition 2.3 states $\mathcal{CO}(K) \subseteq \mathcal{AC}(K)$. Using the properties of translation invariant means we can see that the set $\mathcal{AC}(K)$ of functions in $L^\infty(K)$ such that the mean is unique is a translation invariant closed linear subspace of $L^\infty(K)$.

Now we will look at an important subclass of $\mathcal{AC}(K)$, the set of weakly almost periodic functions $\mathcal{WAP}(K)$:

Definition 2.5. *Let K be a commutative hypergroup. Define*

$$\mathcal{UC}(K) := \{f \in L^\infty(K) : x \mapsto T_x f \text{ is continuous w.r.t. the norm topology of } L^\infty(K)\}$$

$$\begin{aligned} &\mathcal{WAP}(K) := \\ &\quad \{f \in L^\infty(K) : \{T_x f : x \in K\} \text{ is relatively weakly compact in } L^\infty(K)\} \cap \mathcal{UC}(K) \end{aligned}$$

An element $f \in \mathcal{WAP}(K)$ is called weakly almost periodic.

That $\mathcal{WAP}(K)$ is indeed a subclass of $\mathcal{AC}(K)$ is a consequence of the following Proposition which is a classical result showing uniqueness of the mean for the space of weakly almost periodical functions on a commutative hypergroup (which utilizes Proposition 2.3). A proof has been given by Wolfenstetter [31]:

Theorem 2.6. *Let K be a commutative hypergroup. Then there exists a unique translation invariant mean m on $\mathcal{WAP}(K)$ satisfying* *(i)* $m(f) = m(T_x f)$ *for all* $x \in K$, $f \in \mathcal{WAP}(K)$

(ii) $m(\mathbf{1}) = 1$

(iii) $f \geq 0 \Rightarrow m(f) \geq 0$ *for all* $f \in \mathcal{WAP}(K)$

(iv) $\|m\| = 1$

(v) $m(\overline{f}) = \overline{m(f)}$ *for all* $f \in \mathcal{WAP}(K)$

Proof. The existence is guaranteed by Dixmier's [7] result. For $f \in \mathcal{WAP}(K)$ there lies a constant function in the norm closed convex hull of f (again cf. [7]). With Proposition 2.3 the proof is then finished. □

Looking more closely at the proof of Theorem 2.6 it states that the class of weakly almost periodic functions lies in $\mathcal{CO}(K)$, i.e. $\mathcal{WAP}(K) \subseteq \mathcal{CO}(K)$.

Remark 2.7. *In contrast to the group case,* $\mathcal{WAP}(K)$ *fails to be an algebra in general. For a counterexample, see [31]. There Wolfenstetter shows the existence of a weakly almost periodic sequence* h *on a polynomial hypergroup such that* $h^2 \notin \mathcal{WAP}(\mathbb{N}_0)$*.*

For characters χ the mean is unique and we can even specify the value $m(\chi)$:

Corollary 2.8. *Let* m *be a translation invariant mean on* $L^\infty(K)$*. For every character* $\chi \neq \mathbf{1}$ *on* K *we get* $m(\chi) = 0$*.*

Proof. Every character χ satisfies $T_x\chi = \chi(x)\chi$. So if $\chi \neq \mathbf{1}$ there exists $x \in K$ such that $\chi(x) \neq 1$ and thus

$$m(\chi) = m(T_x\chi) = m(\chi(x)\chi) = \chi(x) \cdot m(\chi)$$

which holds only for $m(\chi) = 0$. □

For the identity character, the mean is obviously 1.

For the remainder of this section, we will study a space larger than $\mathfrak{M}_t(K)$ by dropping the requirement of being a mean, in contrast to $\mathfrak{M}(K)$ where we dropped the additional assumption of translation invariance. In fact we will investigate

$$J(L^\infty(K)) := \{F \in L^\infty(K)^* : T_x F = F \text{ for all } x \in K\}.$$

This is defined via the elements $x \in K$ acting on $L^\infty(K)^*$ by the map $L^\infty(K)^* \to L^\infty(K)^*, F \mapsto T_x F$, where $T_x F(\varphi) := F(T_x\varphi)$ for $\varphi \in L^\infty(K)$. Obviously $J(L^\infty(K))$ is a linear subspace of $L^\infty(K)^*$ and $0 \in J(L^\infty(K))$.

Proposition 2.9. *The linear span* $L := span\{T_x\varphi - \varphi : \varphi \in L^\infty(K), x \in K\}$ *is not dense in* $L^\infty(K)$ *if and only if* $J(L^\infty(K)) \neq \{0\}$*.*

The proof follows the lines of Proposition (2.1) in [24].

As there always exists a translation invariant mean for commutative hypergroups and so $J(L^\infty(K)) \supsetneq \{0\}$, Proposition 2.9 states that L cannot be dense in $L^\infty(K)$.

The relation between $J(L^\infty(K))$ and $\mathfrak{M}_t(K)$ is not only that $\mathfrak{M}_t(K) \subseteq J(L^\infty(K))$ and that $\mathfrak{M}_t(K)$ has the additional requirement of its elements being means:

Proposition 2.10. *The space $J(L^\infty(K))$ is linearly spanned by $\mathfrak{M}_t(K)$.*

Proof. The proof follows the lines of Proposition (2.2) in [24].
We identify $L^\infty(K)^*$ with $M(\Phi(K))$ by the Riesz Representation Theorem, where $\Phi(K)$ is the carrier space of $L^\infty(K)$. Let ν be a nonzero element of $J(L^\infty(K))$. We decompose

$$\nu = (\nu_1^+ - \nu_1^-) + i(\nu_2^+ - \nu_2^-)$$

where ν_1 and ν_2 are the real and imaginary parts of ν and ν_i^+ and ν_i^- are the positive and negative variations of ν_i. It suffices to show that each nonzero ν_i^+, ν_i^- is a multiple of an element in $\mathfrak{M}_t(K)$. Without loss of generality (taking scalar multiples if necessary) we assume that $\nu_1^+ \neq 0$. We will show that $\nu_1^+ \in J(L^\infty(K))$ as then $\frac{\nu_1^+}{\|\nu_1^+\|} \in \mathfrak{M}_t(K)$. Now if $\varphi \in L^\infty(K)$ real-valued and $x \in K$, then

$$\nu_1(\varphi - T_x\varphi) = \mathrm{Re}\nu(\varphi - T_x\varphi) = 0$$

so that $\nu_1 \in J(L^\infty(K))$. Furthermore, $\nu_1 = T_x\nu_1 = T_x\nu_1^- T_x\nu_1^-$ with $\nu_1^+, \nu_1^- \geq 0$. By the minimum property of the Jordan decomposition it follows that $T_x\nu_1^+ \geq \nu_1^+, T_x\nu_1^- \geq \nu_1^-$. This leads to

$$\|T_x\nu_1^+ - \nu_1^+\| = (T_x\nu_1^+ - \nu_1^+)(\mathbf{1}) = 0$$

since $(T_x\nu_1^+ - \nu_1^+)$ is a positive functional on $L^\infty(K)$. Thus $T_x\nu_1^+ = \nu_1^+$, i.e. $\nu_1^+ \in J(L^\infty(K))$. □

2.2 Strong Amenability

After studying amenability of commutative hypergroups one can think of stronger versions of amenability as a replacement of the group structure in the classical case, where one has $m(f \cdot T_xg) = m(T_{x^{-1}}f \cdot g)$ for translation invariant means. As there are no proper inverses on hypergroups, one can look for means with a certain additional property that allows a similar operation as the one above for groups with the inverse replaced by the involution. This leads to our definition of strongly translation invariant means:

Definition 2.11. *A mean $m \in \mathfrak{M}(K)$ is called strongly translation invariant if*

$$m((T_y\varphi)\psi) = m(\varphi(T_{\tilde{y}}\psi)) \text{ for all } y \in K, \varphi, \psi \in L^\infty(K).$$

Remark 2.12. *(a) With $\psi = \mathbf{1}$ in Definition 2.11 it is obvious that m is translation invariant whenever m is strongly translation invariant.*

(b) *Let m be a strongly translation invariant mean. For characters $\alpha, \beta \in \widehat{K}, \alpha \neq \beta$ we get $m(\alpha \cdot \overline{\beta}) = 0$. To see that, choose $y \in K$ such that $\alpha(y) \neq \beta(y)$. Then*

$$\alpha(y)m(\alpha \cdot \overline{\beta}) = m(T_y\alpha \cdot \overline{\beta}) = m(\alpha \cdot T_{\tilde{y}}\overline{\beta}) = \beta(y)m(\alpha \cdot \overline{\beta}),$$

hence $m(\alpha \cdot \overline{\beta}) = 0$.

Next we will introduce two Reiter-type conditions and investigate their relation to the existence of strongly translation invariant means.

Definition 2.13. *A hypergroup is said to satisfy the* **uniform finite strong Reiter's condition $\mathbf{P_1}$ ($\mathbf{UFP_1}$)** *if for all $\varepsilon > 0, F \subset K$ finite there exists $g \in L^1(K)$ such that*

(i) $\widehat{g}(\mathbf{1}) = 1$

(ii) $\|g\|_1 = 1$

(iii) $\|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon$ *for all* $y \in F, \varphi \in L^\infty(K)$ *with* $\|\varphi\|_\infty \leq 1$.

Remark 2.14. *(iii) in Definition 2.13 can also be stated as*

$$(iii')\|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon\|\varphi\|_\infty \text{ for all } y \in F, \varphi \in L^\infty(K).$$

Theorem 2.15. *There exists a strongly translation invariant mean on K if K satisfies Reiter's condition (UFP_1).*

Proof. Assume that Reiter's condition (UFP_1) holds, i.e. for every $\varepsilon > 0, F \subseteq K$ finite there exists $g \in L^1(K)$ with $\widehat{g}(\mathbf{1}) = 1 = \|g\|_1$ such that $\|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon$ for all $y \in F, \varphi \in L^\infty(K), \|\varphi\|_\infty \leq 1$. We define functionals $m_{\varepsilon,F} \in (L^\infty(K))^*$ by

$$m_{\varepsilon,F}(\psi) := \int_K g(x)\psi(x)d\mu(x) \text{ for } \psi \in L^\infty(K).$$

They satisfy $m_{\varepsilon,F}(\mathbf{1}) = \|m_{\varepsilon,F}\|_1 = \widehat{g}(\mathbf{1}) = \|g\|_1 = 1$, and so the functionals $m_{\varepsilon,F}$ are elements of a weak*-compact subset of $L^\infty(K)^*$. For each $\psi \in L^\infty(K), \|\psi\|_\infty \leq 1$ we get

$$m_{\varepsilon,F}(\varphi T_{\tilde{y}}\psi) = \int_K g(x)\varphi(x)T_{\tilde{y}}\psi(x)d\mu(x) = \int_K T_y(g\varphi)(x)\psi(x)d\mu(x).$$

Hence

$$\begin{aligned}
|m_{\varepsilon,F}(\varphi T_{\tilde{y}}\psi) - m_{\varepsilon,F}(\psi T_y\varphi)| &= \left|\int_K T_y(g\varphi)(x)\psi(x) - g(x)T_y\varphi(x)\psi(x)d\mu(x)\right| \\
&\leq \int_K |\psi(x)| \cdot |T_y(\varphi g)(x) - T_y\varphi(x)g(x)|d\mu(x) \\
&\leq \|\psi\|_\infty \cdot \|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon\|\psi\|_\infty \leq \varepsilon.
\end{aligned}$$

Defining a relation by

$$(\varepsilon_1, F_1) \prec (\varepsilon_2, F_2) :\Longleftrightarrow \varepsilon_2 \leq \varepsilon_1, F_1 \subseteq F_2$$

we get a partial order, and with respect to this partial order the functionals $m_{\varepsilon,F}$ form a net. By the compactness this net has an accumulation point $m \in (L^\infty(K))^*$ satisfying $m(\mathbf{1}) = 1 = \|m\|$ and

$$m(\varphi T_{\tilde{y}}\psi) = m(\psi T_y \varphi) \text{ for all } \varphi, \psi \in L^\infty(K) \text{ with } \|\varphi\|_\infty \le 1, \|\psi\|_\infty \le 1$$

and all $y \in K$. By the linearity of m this holds for all $\varphi, \psi \in L^\infty(K)$, and this proves that K has a strongly translation invariant mean. □

The converse of Theorem 2.15 has not yet been proved but it is possible to get to a different variant of the strong Reiter's condition which we call local instead of uniform as there exists a (possibly different) $g = g_\varphi$ for every φ whereas in the uniform condition (UFP_1) there is one g that is the same for all $\varphi \in l^\infty$.

Definition 2.16. *A hypergroup is said to satisfy the* **local finite strong Reiter's condition $\mathbf{P_1}$ ($\mathbf{LFP_1}$)** *if for all $\varepsilon > 0, F \subset K$ finite and for all $\varphi \in L^\infty(K)$ there exists $g = g_\varphi \in L^1(K)$ such that*

(i) $\widehat{g}(1) = 1$

(ii) $\|g\|_1 = 1$

(iii) $\|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon$ *for all* $y \in F$.

Theorem 2.17. *If there exists a strongly translation invariant mean on K then K satisfies Reiter's condition (LFP_1).*

Proof. Assume there exists a strongly translation invariant mean m on K. By Goldstine's Theorem [8, p.424] the embedding of the unit ball $B \subseteq L^1(K)$ into $L^\infty(K)^*$ is dense in the unit ball $B^{**} \subseteq L^\infty(K)^*$ with respect to the weak*-topology. Hence there exists a net $(f_j)_{j\in I}$ of functions in B such that

$$\int_K f_j(x)\psi(x)d\mu(x) \to m(\psi) \text{ for all } \psi \in L^\infty(K).$$

In particular, $\widehat{f_j}(\mathbf{1}) \to m(\mathbf{1}) = 1$. Since m is positive we can assume that $f_j \ge 0, \|f_j\|_1 = 1$. For any $y \in K$ and $\varphi \in L^\infty(K), \|\varphi\|_\infty \le 1$ we have

$$\int_K T_y(\varphi f_j)(x)\psi(x)d\mu(x) = \int_K f_j(x)\varphi(x)T_{\tilde{y}}\psi(x)d\mu(x) \to m(\varphi T_{\tilde{y}}\psi) = m(\psi T_y\varphi)$$

and

$$\int_K f_j(x)T_y\varphi(x)\psi(x)d\mu(x) \to m(\psi T_y \varphi).$$

Hence

$$\int_K (T_y(\varphi f_j)(x) - (f_jT_y\varphi)(x))\,\psi(x)d\mu(x) \to 0 \text{ for all } \psi \in L^\infty(K).$$

Given $y_1, \ldots, y_m \in K$ and $\varphi \in l^\infty$ define $G_{k,j} := T_{y_k}(\varphi f_j) - f_jT_{y_k}\varphi$. Then the net defined by $G_j := (G_{1,j}, \ldots, G_{m,j}) \in L^1(K) \times \ldots \times L^1(K)$ converges to $\mathbf{0}$ in the product space with respect to the weak topology. Then there exists a convex combination of elements G_j that converges in the norm of the product space $L^1(K) \times \ldots \times L^1(K)$ to $\mathbf{0}$ [8, p.422]. Thus for each $\varepsilon > 0$, $F \subseteq K$ finite and $\varphi \in L^\infty(K)$ there exists $g \in L^1(K)$ as a convex combination of functions f_j satisfying

$$\widehat{g}(\mathbf{1}) = 1, \|g\|_1 = 1 \text{ and } \|T_{y_k}(\varphi g) - g T_{y_k}\varphi\|_1 < \varepsilon.$$

□

As an example to hypergroups admitting strongly translation invariant means we will take certain polynomial hypergroups, see Proposition 3.15.

3 Translation invariant means on polynomial hypergroups

For sequences $(x_n)_{n\in\mathbb{N}}$ the existence of means on l^∞ invariant with respect to the left shift has been shown via Banach limits [8]. In the corresponding proof, Hahn-Banach theorem is used with $p((x_n)_{n\in\mathbb{N}}) = \limsup\limits_{n\to\infty} x_n$ as translation invariant subadditive function and the resulting continuation of $l(x) = \lim\limits_{n\to\infty} x_n$ for $x \in \{y \in l^\infty : (y_n)_{n\in\mathbb{N}} \text{ converges}\}$ is called a Banach limit. This idea does not work for polynomial hypergroups, as in general $\limsup\limits_{n\to\infty} \varphi_n \neq \limsup\limits_{n\to\infty} (T_k\varphi)_n$ for the hypergroup translation T_k which is a convolution instead of a mere shift. The existence of a translation invariant mean on polynomial hypergroups has already been shown more generally in Chapter 2, but by using the construction at the end of Section 2.1 there is a proof of the existence of translation invariant means on polynomial hypergroups that does not need the Markov-Kakutani fixed point theorem. First of all we define $L := span\{T_n\varphi - \varphi : \varphi \in l^\infty, n \in \mathbb{N}_0\}$ as in Proposition 2.9. We can simplify that to

Lemma 3.1. $L = span\{T_1\varphi - \varphi : \varphi \in l^\infty\}$

Proof. We use induction on n to show that $T_n\varphi - \varphi \in span\{T_1\psi - \psi : \psi \in l^\infty\}$ for every $\varphi \in l^\infty$. For $n = 0, 1$ this is obviously satisfied. Now assume that $T_k\sigma - \sigma \in span\{T_1\psi - \psi : \psi \in l^\infty\}$ for all $k = 0, \ldots, n$ and every $\sigma \in l^\infty$. With

$$T_{n+1} = \frac{1}{a_n} T_1 \circ T_n - \frac{b_n}{a_n} T_n - \frac{c_n}{a_n} T_{n-1} \text{ for } n \geq 1$$

we get for every $\varphi \in l^\infty$

$$\begin{aligned} T_{n+1}\varphi - \varphi &= T_1\left(\frac{T_n\varphi}{a_n}\right) - \frac{T_n\varphi}{a_n} + \frac{T_n\varphi}{a_n} - \frac{b_n T_n\varphi}{a_n} - \frac{c_n T_{n-1}\varphi}{a_n} - \frac{a_n}{a_n}\varphi \\ &= T_1\psi - \psi + T_n\sigma - T_{n-1}\tau - (\sigma - \tau) \end{aligned}$$

where

$$\psi = \frac{T_n\varphi}{a_n}, \sigma = \frac{1 - b_n}{a_n}\varphi, \tau = \frac{c_n\varphi}{a_n}$$

which finishes the proof. □

We can now further simplify by observing that $\{T_1\varphi - \varphi : \varphi \in l^\infty\}$ is a linear space as every $\sum\limits_{k=1}^{n} \alpha_k(T_1\varphi_k - \varphi_k) \in L$ is just $T_1(\sum\limits_{k=1}^{n} \alpha_k\varphi_k) - \sum\limits_{k=1}^{n} \alpha_k\varphi_k$. With that in mind we can state the following result:

Theorem 3.2. *On polynomial hypergroups there exists a translation invariant mean.*

Proof. We first prove that $\|T_1\varphi - \varphi - \mathbf{1}\|_\infty \geq 1$ for all $\varphi \in l^\infty$.
Observe that if φ is a convergent sequence, then $T_1\varphi(n) - \varphi(n) \to 0$ and so $\|T_1\varphi - \varphi - \mathbf{1}\|_\infty \geq 1$. We see that

$$\begin{aligned} T_1\varphi(n) - \varphi(n) &= a_n\varphi(n+1) + b_n\varphi(n) + c_n\varphi(n) - (a_n + b_n + c_n)\varphi(n) \\ &= a_n(\varphi(n+1) - \varphi(n)) + c_n(\varphi(n-1) - \varphi(n)) \end{aligned}$$

If there exists $n \in \mathbb{N}$ such that both $\varphi(n+1) - \varphi(n)$ and $\varphi(n+1) - \varphi(n)$ are negative, the first part of the proof is finished as then $\|T_1\varphi - \varphi - \mathbf{1}\|_\infty \geq 1$. If there exists no such n, then either φ is decreasing, increasing or there exists n_0 such that φ is decreasing up to n_0 and from then on increasing. In all three cases φ is eventually monotone and as a bounded sequence it converges. Thus the first statement is proven.

As L is thus not dense in l^∞, by Proposition 2.9 we get that $J(l^\infty) \neq \{0\}$. As $J(l^\infty)$ is linearly spanned by $\mathfrak{M}_t(\mathbb{N}_0)$ by Proposition 2.10, there must be a nonzero element in $\mathfrak{M}_t(\mathbb{N}_0)$ (see the proof of Proposition 2.10) and thus there exists a translation invariant mean. □

Utilizing the specific structure of hypergroups derived from sequences of orthogonal polynomials we can get some results that do not hold on general commutative hypergroups. Especially the 3-term recurrence relation is a very helpful tool in many proofs in this chapter. The first application of this idea simplifies proving translation invariance for a mean such that one only has to check invariance with respect to T_1:

Proposition 3.3. *A mean $m \in \mathfrak{M}(\mathbb{N}_0)$ is translation invariant if $m(T_1\varphi) = m(\varphi)$ for all $\varphi \in l^\infty(\mathbb{N}_0)$.*

Proof. We use induction on n. Assume that $m(T_k\varphi) = m(\varphi)$ for $k = 0, 1, \ldots, n$. Then

$$m(T_{n+1}\varphi) = \frac{1}{a_n} m(T_1 \circ T_n\varphi) - \frac{b_n}{a_n} m(T_n\varphi) - \frac{c_n}{a_n} m(T_{n-1}\varphi) = m(\varphi)$$

where a_n, b_n, c_n are the coefficients of the three-term recurrence relation of the underlying family of orthogonal polynomials. □

We also prove the following Lemma which is helpful for representing the powers of the translation operator T_1 for polynomial hypergroups in a convenient way:

Lemma 3.4. *T_1^n can be written as a convex combination of the form*

$$T_1^n = \sum_{k=0}^{n} \alpha_k T_k$$

where $\sum_k \alpha_k = 1$ and $\alpha_k \geq 0$ for all k.

Proof. Use the 3-term recurrence relation to find inductively:

$$T_1T_1 = a_1T_2 + b_1T_1 + c_1T_0 \text{ and then}$$

$$\begin{aligned} T_1^{n+1} &= T_1T_1^n = T_1\left(\sum_{k=0}^{n}\alpha_kT_k\right) \\ &= \sum_{k=0}^{n}\alpha_kT_1T_k = \sum_{k=0}^{n}\alpha_k(a_kT_{k+1} + b_kT_k + c_kT_{k-1}). \end{aligned}$$

This is a convex combination of $\{T_k : 0 \le k \le n+1\}$. □

With Proposition 2.3 we can also find the following criterion for the uniqueness of the mean for polynomial hypergroups:

Corollary 3.5. *Let $\varphi \in l^\infty$ such that $T_1^n\varphi$ converges uniformly to $\psi \in l^\infty$ for $n \to \infty$. Then $m(\varphi)$ is uniquely determined.*

Proof.

$$\psi = \lim_{n\to\infty} T_1^n\varphi = \lim_{n\to\infty} T_1^{n+1}\varphi = T_1 \lim_{n\to\infty} T_1^n\varphi = T_1\psi$$

Since the only T_1-invariant functions on l^∞ are constants and, by Lemma 3.4, $\psi \in \overline{co\{T_n\varphi : n \in \mathbb{N}_0\}}^\infty$ the proof is finished. □

This sequence does not converge for all $\varphi \in l^\infty$ on all polynomial hypergroups as the following counterexample shows:

Example 3.6. *Consider the polynomial hypergroup generated by the Chebyshev polynomials of the first kind and $\varphi := (1,0,1,0,1,0,\ldots)$. $T_1\varphi = (0,1,0,1,\ldots)$ and $T_1^2\varphi = \varphi$, so the sequence $(T_1^n\varphi)_{n\in\mathbb{N}_0}$ does not converge but oscillate between two elements. Incidentally the mean for φ is still unique as $\frac{1}{2}\varphi + \frac{1}{2}T_1\varphi = (\frac{1}{2},\frac{1}{2},\ldots)$ and so $m(\varphi) = \frac{1}{2}m(\varphi) + \frac{1}{2}m(T_1\varphi) = m((\frac{1}{2},\frac{1}{2},\ldots)) = \frac{1}{2}$.*

In fact it is not possible on a polynomial hypergroup that means coincide for all $\varphi \in l^\infty$ as we will see in Chapter 4. But we will try to find a preferably large subset of functions in l^∞ such that $m(\varphi)$ is given by φ alone, not depending on the choice of m.

For convergent sequences on polynomial hypergroups we can prove the following strong result:

Proposition 3.7. *Let $\varphi \in l^\infty$ such that $\lim_{n\to\infty}\varphi_n =: c$. Then $m(\varphi) = c$.*

Proof. Due to the linearity of m it is sufficient to show that $m(\varphi) = 0$ for $\varphi \in c_0$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}_0$ such that $|\varphi_n| \le \varepsilon$ for $n \ge N$. Define $\varphi^1 := (\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{N-1}, 0, \ldots)$ and $\varphi^2 := (0,\ldots,0,\varphi_N,\varphi_{N+1},\ldots)$. Again using the linearity of m we get

$$m(\varphi^1) = \varphi_0 m(\varepsilon_0) + \ldots + \varphi_{N-1}m(\varepsilon_{N-1}) = \sum_{k=0}^{N-1}\varphi_k m(\varepsilon_0)$$

as $\varepsilon_k = T_k\varepsilon_0$, and $m(\varphi^2) \le \|\varphi^2\|_\infty \le \varepsilon$.
So if $m(\varepsilon_0) = 0$ the proof is finished. Assume $m(\varepsilon_0) = c > 0$. Choose $k \in \mathbb{N}$ such that $k > \frac{1}{c}$. Then $m(\underbrace{1,\ldots,1}_{k},0,\ldots) = k \cdot c > 1$, a contradiction to $m(\underbrace{1,\ldots,1}_{k},0,\ldots) \le m(\mathbf{1}) = 1$ □

On polynomial hypergroups, we can get stricter bounds for means than the norm $\|\varphi\|_\infty$ for general hypergroups as in Remark 2.2:

Proposition 3.8. *Let m be a translation invariant mean and $\varphi = (\varphi_n)_{n\in\mathbb{N}_0} \in l^\infty$. Then*

$$\liminf_{n\to\infty} \varphi_n \le m(\varphi) \le \limsup_{n\to\infty} \varphi_n$$

Proof. By Proposition 3.7 we get

$$m(\varphi) = m(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}, 0, \ldots) + m(0, \ldots, 0, \varphi_n, \varphi_{n+1}, \ldots) \ge 0 + \inf_{k\ge n} \varphi_k.$$

Since that holds for each $n \in \mathbb{N}_0$, $m(\varphi) \ge \liminf\limits_{n\to\infty} \varphi_n$. Analogously for the other inequality. □

In Chapter 2 we have studied means in general, and criteria for when the mean is unique. In the present chapter we want to find explicit representations for means, especially when it is unique. Then a formula to calculate $m(\varphi)$ might be helpful for practical applications. We will restrict our attention to the case of polynomial hypergroups and give criteria for when a certain construction using "summing sequences", an analogue to Følner sequences [11] on groups, works.

Recall that a polynomial hypergroup satisfies condition (H) if and only if

$$\lim_{n\to\infty} \frac{h(n)}{\sum\limits_{k=0}^{n} h(k)} = 0.$$

3.1 Summing sequences

A basic tool which we will use in constructing representations of means are so-called summing sequences:

Definition 3.9. *A sequence $(A_n)_{n\in\mathbb{N}_0}$ where $A_n \subseteq \mathbb{N}_0$ for all $n \in \mathbb{N}_0$ is called* **summing sequence** *on the polynomial hypergroup $\mathbb{N}_0$ if it satisfies*

(i) $A_n \subseteq A_{n+1}$ *for all* $n \in \mathbb{N}_0$

(ii) $\bigcup\limits_{n\in\mathbb{N}_0} A_n = \mathbb{N}_0$

(iii) $h(A_n) < \infty$ *for all* $n \in \mathbb{N}_0$

(iv) $\lim_{n\to\infty} \frac{h(T_kA_n\Delta A_n)}{h(A_n)} = 0$ *for all* $k \in \mathbb{N}_0$

Very much like Følner sequences on groups, the sets in these summing sequences are ascending, eventually cover all of $\mathbb{N}_0$, are compact and the symmetric difference between a set A_n and its translate T_kA_n is "small" compared to the size of the original set A_n for increasing n. A natural candidate for a summing sequence is $S_n := \{0, 1, \dots, n\}$. We will call that specific choice the **canonical sequence**.

Theorem 3.10. *The canonical sequence* $(S_n)_{n\in\mathbb{N}_0}$ *is a summing sequence if and only if the hypergroup satisfies condition (H).*

Proof. As $\{n+k\} \subseteq T_kS_n\Delta S_n \subseteq \{n+1, \dots, n+k\}$ for $n \geq k$, we get

$$\frac{h(n+k)}{h(S_n)} \leq \frac{h(T_kS_n\Delta S_n)}{h(S_n)} \leq \frac{h(n+1)+\ldots+h(n+k)}{h(S_n)}.$$

If (H) holds, Proposition 1.13 implies that $(S_n)_{n\in\mathbb{N}_0}$ is a summing sequence. Conversely, if $(S_n)_{\mathbb{N}_0}$ is a summing sequence, the first inequality (with k=1) yields property (H). □

The above result is a nice characterization of when (i.e. for which hypergroups) the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ is a summing sequence, as the important structural condition (H) appears. There are other examples of summing sequences, which can be obtained as follows:

Proposition 3.11. *Let (H) be satisfied and* $(A_n)_{n\in\mathbb{N}_0}$ *a sequence of sets such that*

(i) $A_n \subseteq A_{n+1}$ *for all* $n \in \mathbb{N}_0$

(ii) $\bigcup_{n\in\mathbb{N}_0} A_n = \mathbb{N}_0$

(iii) $A_n = S_n \setminus I_n$ *where* $(S_n)_{n\in\mathbb{N}_0}$ *is the canonical sequence and* $|I_n| \leq C$ *for all* $n \in \mathbb{N}_0$ *for some* $C > 0$

If the Haar measure h is nondecreasing, $(A_n)_{n\in\mathbb{N}_0}$ *is a summing sequence.*

Proof. Because of

$$\begin{aligned} h(T_kA_n\Delta A_n) &\leq h(T_kS_n\Delta S_n) + h(T_kI_n) + h(I_n) \\ &\leq h(T_kS_n\Delta S_n) + (2k+2)C \cdot h(n+k) \end{aligned}$$

and Proposition 1.13 and Theorem 3.10, $(A_n)_{n\in\mathbb{N}_0}$ is a summing sequence. □

The sequences $(A_n)_{n\in\mathbb{N}_0}$ in Proposition 3.11 are very similar to the "natural" summing sequence $(S_n)_{n\in\mathbb{N}_0}$. In contrast we will now see an example of a sequence $(A_n)_{n\in\mathbb{N}_0}$ that satisfies (i)-(iii) of Definition 3.9 but not condition (iv). For the simplest case take the polynomial hypergroup induced by the Chebyshev polynomials of the first kind, i.e. $h(0) = 1, h(n) = 2$ and $T_1\{n\} = \{n-1, n+1\}$ for all $n \in \mathbb{N}$. Define $A_n := \{0, 1, \dots, 2n\} \cup \{2n+1, 2n+3, \dots, 4n-1\}$. Then $T_1A_n = \{0, 1, \dots, 2n, 2n+1\} \cup \{2n+2, 2n+4, \dots, 4n\}$, and $T_1A_n\Delta A_n = \{2n+2, 2n+3, \dots, 4n\}$. Then $h(A_n) = 6n+1$

and $h(T_1A_n\Delta A_n) = 4n-2$, and so $\lim\limits_{n\to\infty} \frac{h(T_1A_n\Delta A_n)}{h(A_n)} = \frac{2}{3}$, i.e. $(A_n)_{n\in\mathbb{N}_0}$ is not a summing sequence.
We can also give an example of a summing sequence which is not "almost" $(S_n)_{n\in\mathbb{N}_0}$ like in Proposition 3.11: Let (H) be satisfied and let the Haar measure be nondecreasing. Define a sequence of sets by $A_n := \{0,1,\dots,3^n\} \cup \{2\cdot 3^n+1, 2\cdot 3^n+2,\dots,3^{n+1}\}$. (i)-(iii) of Definition 3.9 are again obviously satisfied and since the Haar measure is nondecreasing we get $h(T_1A_n\Delta A_n) \le 3\cdot h(3^{n+1}+1)$ and by Proposition 1.13 and using $2h(A_n) \ge h(S_{3^n})$ we get that $(A_n)_{n\in\mathbb{N}_0}$ is a summing sequence.
So far we have seen some examples of summing sequences and some sequences of sets which are not summing sequences. The following Proposition gives an answer to the question of whether there always exists a summing sequence, for any given polynomial hypergroup. It is again related to condition (H), but in a kind of strong form of its converse:

Proposition 3.12. *If* $\frac{h(n)}{\sum\limits_{k=0}^{n} h(k)} \ge C > 0$ *for all* $n \in \mathbb{N}_0$ *there are no summing sequences.*

Proof. Let $(A_n)_{n\in\mathbb{N}_0}$ be a sequence of sets that satisfies (i)-(iii) of Definition 3.9 and $m_n := \max A_n$. Then $m_n \le m_{n+1}$, $\lim\limits_{n\to\infty} m_n = \infty$ and $m_n+1 \in T_1A_n\Delta A_n$. Therefore

$$\frac{h(T_1A_n\Delta A_n)}{h(A_n)} \ge \frac{h(m_n+1)}{h(A_n)} \ge \frac{h(m_n+1)}{\sum\limits_{k=0}^{m_n+1} h(k)} \ge C > 0$$

and so (iv) of Definition 3.9 cannot hold. □

In the case that summing sequences exist we can get the following helpful result which yields translation invariance in the limit of a partial summation, very much like in the group case where one works with Følner sequences. We will even show strong translation invariance, a concept that has been introduced in Section 2.2.

Proposition 3.13. *Let* $(A_n)_{n\in\mathbb{N}_0}$ *be a summing sequence and let* $g_n : l^\infty(\mathbb{N}_0) \to \mathbb{C}$, $g_n(\varphi) := \frac{1}{h(A_n)} \sum\limits_{k\in A_n} \varphi(k)h(k)$. *Then*

$$\lim_{n\to\infty} |g_n(\psi T_s\varphi) - g_n(\varphi T_s\psi)| = 0 \text{ for all } \varphi,\psi \in l^\infty(\mathbb{N}_0), s\in\mathbb{N}.$$

Proof. We have

$$\begin{aligned}
g_n(\varphi T_s\psi) &= \frac{1}{h(A_n)}\sum_{k\in A_n}\varphi(k)T_s\psi(k)h(k)\\
&= \frac{1}{h(A_n)}\sum_{k\in A_n}\varphi(k)\sum_{m\in T_s\{k\}}g(s,k;m)\psi(m)h(k)\\
&= \frac{1}{h(A_n)}\sum_{k\in A_n}\varphi(k)\sum_{m\in T_s\{k\}}g(s,m;k)\psi(m)h(m)\\
&= \frac{1}{h(A_n)}\sum_{m\in T_sA_n}\left(\sum_{k\in A_n}g(s,m;k)\varphi(k)\right)\psi(m)h(m)\\
&= \frac{1}{h(A_n)}\sum_{m\in T_sA_n}T_s\varphi(m)\psi(m)h(m)\\
&\quad -\frac{1}{h(A_n)}\sum_{m\in T_sA_n}\left(T_s\varphi(m)-\sum_{k\in A_n}g(s,m;k)\varphi(k)\right)\psi(m)h(m)
\end{aligned}$$

and thus

$$\begin{aligned}
&|g_n(\psi T_s\varphi)-g_n(\varphi T_s\psi)|\\
&\leq \frac{1}{h(A_n)}\left|\sum_{k\in T_sA_n\Delta A_n}T_s\varphi(k)\psi(k)h(k)\right|\\
&\quad +\frac{1}{h(A_n)}\left|\sum_{m\in T_sA_n}\left(T_s\varphi(m)-\sum_{k\in A_n}g(s,m;k)\varphi(k)\right)\psi(m)h(m)\right|
\end{aligned}$$

For the first term we get

$$\frac{1}{h(A_n)}\left|\sum_{k\in T_sA_n\Delta A_n}T_s\varphi(k)\psi(k)h(k)\right|\leq \|\varphi\|_\infty\|\psi\|_\infty\frac{h(T_sA_n\Delta A_n)}{h(A_n)}$$

and for the second one

$$
\begin{aligned}
&\frac{1}{h(A_n)}\left|\sum_{m\in T_sA_n}\left(T_s\varphi(m)-\sum_{k\in A_n}g(s,m;k)\varphi(k)\right)\psi(m)h(m)\right|\\
&= \frac{1}{h(A_n)}\left|\sum_{m\in T_sA_n}\left(\sum_{k\in T_sT_sA_n\setminus A_n}g(s,m;k)\varphi(k)\right)\psi(m)h(m)\right|\\
&\leq \frac{1}{h(A_n)}\|\varphi\|_\infty\|\psi\|_\infty\left|\sum_{k\in T_sT_sA_n\setminus A_n}\ \sum_{m\in T_sA_n}g(s,m;k)h(m)\right|\\
&= \frac{1}{h(A_n)}\|\varphi\|_\infty\|\psi\|_\infty\left|\sum_{k\in T_sT_sA_n\setminus A_n}\ \sum_{m\in T_sA_n}g(s,k;m)h(k)\right|\\
&= \frac{1}{h(A_n)}\|\varphi\|_\infty\|\psi\|_\infty\left|\sum_{k\in T_sT_sA_n\setminus A_n}h(k)\left(\sum_{m\in T_sA_n}g(s,k;m)\right)\right|\\
&\leq \frac{1}{h(A_n)}\|\varphi\|_\infty\|\psi\|_\infty\left|\sum_{k\in T_sT_sA_n\setminus A_n}h(k)\right|\\
&\leq \|\varphi\|_\infty\|\psi\|_\infty\sum_{j=0}^{2s}\frac{h(T_jA_n\Delta A_n)}{h(A_n)}
\end{aligned}
$$

The last inequality holds because of

$$T_sT_sA_n \subseteq \bigcup_{k\in A_n}\{|k-2s|,\dots,k+2s\} \subseteq \bigcup_{k\in A_n}\bigcup_{j=0}^{2s}T_j\{k\} = \bigcup_{j=0}^{2s}T_jA_n.$$

As $(A_n)_{n\in\mathbb{N}_0}$ is a summing sequence the proof is finished. □

As we have already seen, for some classes of functions in l^∞ 'the' mean is unique, i.e. all means give the same value, applied to such a function, e.g. for weakly almost periodic functions (Theorem 2.6) or sequences for which their limit exists (Proposition 3.7). For these functions φ, the mean can be calculated, on hypergroups admitting summing sequences, by taking the limit $\lim_{n\to\infty} g_n(\varphi)$, where g_n are as in Proposition 3.13:

Theorem 3.14. *Let $(A_n)_{n\in\mathbb{N}_0}$ be a summing sequence on $\mathbb{N}_0$ and let $\varphi\in\mathcal{AC}(\mathbb{N}_0)$. Then the sequence $(\frac{1}{h(A_n)}\sum_{k\in A_n}\varphi(k)h(k))_{n\in\mathbb{N}_0}$ converges and the unique translation invariant mean on $\mathcal{AC}(\mathbb{N}_0)$ is given by*

$$M(\varphi)=\lim_{n\to\infty}\frac{1}{h(A_n)}\sum_{k\in A_n}\varphi(k)h(k).$$

This representation is independent of the choice of the summing sequence.

The proof of this result is similar to the proof in the group case, compare e.g. [13].

Proof. It is sufficient to prove the representation for real-valued functions $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$. For such φ, let

$$\mathcal{P}(\varphi) := \limsup_{n\to\infty} g_n(\varphi)$$

where g_n are as defined in Proposition 3.13. $\mathcal{P}$ satisfies $\mathcal{P}(\varphi+\psi) \le \mathcal{P}(\varphi) + \mathcal{P}(\psi)$ and $\mathcal{P}(\alpha\varphi) = \alpha\mathcal{P}(\varphi)$ for $\alpha \ge 0$. By the Hahn-Banach theorem there exists a linear functional M_0 on $\mathcal{AC}_r(\mathbb{N}_0)$ such that

$$-\mathcal{P}(-\varphi) \le M_0(\varphi) \le \mathcal{P}(\varphi) \text{ for all } \varphi \in \mathcal{U}_r(\mathbb{N}_0).$$

With Proposition 3.13 one can see that $\mathcal{P}(T_s\varphi - \varphi) = -\mathcal{P}(-T_s\varphi + \varphi) = 0$ for all $\varphi \in \mathcal{U}_r(\mathbb{N}_0)$ and $s \in \mathbb{N}_0$. This shows that $M_0(T_s\varphi) = M_0(\varphi)$. Also $M_0(\varphi) \ge 0$ whenever $\varphi \ge 0$, $M_0(\mathbf{1}) = 1$ and $\|M_0\| = 1$ are easily seen with the inequality $-\mathcal{P}(-\varphi) \le M_0(\varphi) \le \mathcal{P}(\varphi)$. Thus by the uniqueness of the translation invariant mean on $\mathcal{AC}_r(\mathbb{N}_0)$ we get $M_0(\varphi) = M(\varphi)$ for all $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$.
Assume there is a function $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$ such that

$$-\mathcal{P}(-\varphi) = \liminf_{n\to\infty} g_n(\varphi) < \limsup_{n\to\infty} g_n(\varphi) = \mathcal{P}(\varphi).$$

Then by the Hahn-Banach theorem there would be two functionals M_0 and M_1 which would be equal to M on $\mathcal{AC}(\mathbb{N}_0)$, a contradiction to the assumed uniqueness of the mean. Thus the limit

$$\lim_{n\to\infty} \frac{1}{h(A_n)} \sum_{k\in A_n} \varphi(k)h(k)$$

exists and is equal to M. The argumentation in this proof shows that for every summing sequence this limit exists and is equal to $M(\varphi)$. □

Inspired by Proposition 3.13 where we were able to prove asymptotic **strong** translation invariance when a summing sequence exists we will now show that in that setting the polynomial hypergroup satisfies the unique strong Reiter's condition (UFP_1) and so with the help of Theorem 2.15 this type of hypergroups is strongly amenable.

Proposition 3.15. *A polynomial hypergroup admitting a summing sequence is strongly amenable.*

Proof. We will show that a polynomial hypergroup admitting a summing sequence satisfies Reiter's condition (UFP_1) and use Theorem 2.15. Let $\varepsilon > 0$ and $F := \{y_1, \ldots, y_N\} \subset \mathbb{N}_0$ finite. By assumption there exists a summing sequence $\{A_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \frac{h(T_{y_j}A_k \Delta A_k)}{h(A_k)} = 0$ for all $j = 1, \ldots, N$. Thus there exists $n \in \mathbb{N}$ satisfying $\frac{h(T_{y_j}A_n \Delta A_n)}{h(A_n)} < \frac{\varepsilon}{2}$ for all $j = 1, \ldots, N$. Define $g(k) := \frac{\chi_{A_n}(k)}{h(A_n)}$ and let $y \in F$. Then for all

$\varphi \in L^\infty(\mathbb{N}_0)$ with $\|\varphi\|_\infty \le 1$ we get

$$\|T_y(\varphi g) - gT_y\varphi\|_1 = \sum_{k=0}^{\infty} \left| T_y(\varphi \frac{\chi_{A_n}}{h(A_n)})(k) - \frac{\chi_{A_n}(k)}{h(A_n)} T_y\varphi(k) \right| h(k)$$

$$= \frac{1}{h(A_n)} \sum_{k\in A_n} \left| \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j) - \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j) \right| h(k)$$

$$+ \frac{1}{h(A_n)} \sum_{k\notin A_n} \left| \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j) \right| h(k)$$

For the first term we get

$$\frac{1}{h(A_n)} \sum_{k\in A_n} \left| \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j) - \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j) \right| h(k)$$

$$= \frac{1}{h(A_n)} \sum_{k\in A_n} \left| \sum_{j\in T_y\{k\}\cap A_n} g(k,y;j)\varphi(j) - \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j) \right| h(k)$$

$$= \frac{1}{h(A_n)} \sum_{k\in A_n} \left| \sum_{j\in T_y\{k\}\setminus A_n} g(k,y;j)\varphi(j) \right| h(k)$$

$$= \frac{1}{h(A_n)} \sum_{k\in A_n} \left| \sum_{j\in T_y\{k\}\setminus A_n} g(j,y;k)h(j)\varphi(j) \right|$$

$$\le \frac{1}{h(A_n)} \sum_{k\in A_n} \sum_{j\in T_yA_n\setminus A_n} g(j,y;k)h(j)\,|\varphi(j)|$$

$$= \frac{1}{h(A_n)} \sum_{j\in T_yA_n\setminus A_n} \left(\sum_{k\in A_n} g(j,y;k) \right) h(j)|\varphi(j)|$$

$$\le \frac{h(T_yA_n\Delta A_n)}{h(A_n)} < \frac{\varepsilon}{2}$$

For the second term we note that $\sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j)$ can only be nonzero if there exists $j \in A_n$ such that $j \in T_y\{k\}$, which is equivalent to $k \in T_y\{j\}$. So the outer sum can be taken only over $k \in T_y(A_n)$. Thus we get

$$\frac{1}{h(A_n)} \sum_{k\notin A_n} \left| \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j) \right| h(k)$$

$$= \frac{1}{h(A_n)} \sum_{k\in T_y(A_n)\setminus A_n} \left| \sum_{j\in T_y\{k\}} g(k,y;j)\varphi(j)\chi_{A_n}(j) \right| h(k)$$

$$\le \frac{h(T_yA_n\Delta A_n)}{h(A_n)} < \frac{\varepsilon}{2}$$

Together, $\|T_y(\varphi g) - gT_y\varphi\|_1 < \varepsilon$ and thus Reiter's condition (UFP_1) holds. □

So for polynomial hypergroups admitting summing sequences, e.g. for the ones satisfying condition (H), we have proved the existence of a strongly translation-invariant mean in the sense of Definition 2.11.

3.2 Conditions (F_p)

In the previous section the unique translation invariant mean on a suitable subspace of l^∞ on a polynomial hypergroup was represented by using summing sequences, a construction which is very similar to Følner sequences on groups. As it turns out, weaker conditions are already sufficient to ensure a similar representation of means. Our aim will be to construct invariant means as limits of sequences of functions in $l^p(h)$. To get there, we will study characteristic functions χ_{A_n} of sequences of sets $(A_n)_{n\in\mathbb{N}_0}$ similar to summing sequences. We call the resulting conditions (F_p) where $1 \leq p < \infty$.

Definition 3.16. *A sequence $(A_n)_{n\in\mathbb{N}}$ where $A_n \subseteq \mathbb{N}_0$ for all $n \in \mathbb{N}$ is said to satisfy property $(F_p), 1 \leq p < \infty$, if (i)-(iii) of Definition 3.9 are valid and*

(iv)' $\lim\limits_{n\to\infty} \frac{\|T_k\chi_{A_n} - \chi_{A_n}\|_p^p}{h(A_n)} = 0$ *for all* $k \in \mathbb{N}$.

Remark 3.17. *In the group case condition (iv) of Definition 3.9 coincides with condition (iv)' of Definition 3.16 if $p = 1$, as $\|T_k\chi_{A_n} - \chi_{A_n}\|_1 = h(T_kA_n\Delta A_n)$. In contrast to the group case, however, we have to keep in mind that $T_k\chi_{A_n}$ is no more a characteristic function on a hypergroup.*

In Definition 3.16, (iv)' is required to hold for all $k \in \mathbb{N}$. But due to the three-term recurrence relation this can be relaxed so that it need only be checked for the case $k = 1$:

Lemma 3.18. *For property (F_p) it is sufficient that (iv)' holds only for $k = 1$.*

Proof. The translation operators $T_k : l^p(h) \to l^p(h)$ satisfy the three-term recursion

$$T_1T_k = a_kT_{k+1} + b_kT_k + c_kT_{k-1} \tag{3.1}$$

for $k \in \mathbb{N}$. Moreover, the operator norms fulfill $\|T_k\| \leq 1$ for all $k \in \mathbb{N}$ (and any p). We will apply induction with respect to k:
Assume that (iv) is fulfilled for k and $k-1$. Then

$$\begin{aligned} \|T_kT_1\chi_{A_n} - \chi_{A_n}\|_p &\leq \|T_1T_k\chi_{A_n} - T_1\chi_{A_n}\|_p + \|T_1\chi_{A_n} - \chi_{A_n}\|_p \\ &\leq \|T_k\chi_{A_n} - \chi_{A_n}\|_p + \|T_1\chi_{A_n} - \chi_{A_n}\|_p \end{aligned}$$

and hence

$$\begin{aligned}
\|T_{k+1}\chi_{A_n} - \chi_{A_n}\|_p &= \|\frac{1}{a_k}(T_k T_1 \chi_{A_n} - b_k T_k \chi_{A_n} - c_k T_{k-1}\chi_{A_n}) - \chi_{A_n}\|_p \\
&\leq \|\frac{1}{a_k}(T_k T_1 \chi_{A_n} - \chi_{A_n})\|_p + \|\frac{b_k}{a_k}(T_k \chi_{A_n} - \chi_{A_n})\|_p \\
&\quad + \|\frac{c_k}{a_k}(T_{k-1}\chi_{A_n} - \chi_{A_n})\|_p \\
&\leq \frac{1}{a_k}(\|T_k\chi_{A_n} - \chi_{A_n}\|_p + \|T_1\chi_{A_n} - \chi_{A_n}\|_p) + \frac{b_k}{a_k}\|T_k\chi_{A_n} - \chi_{A_n}\|_p \\
&\quad + \frac{c_k}{a_k}\|T_{k-1}\chi_{A_n} - \chi_{A_n}\|_p.
\end{aligned}$$

Because of the observation $\lim_{n\to\infty} \frac{\|T_k\chi_{A_n}-\chi_{A_n}\|_p^p}{h(A_n)} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{\|T_k\chi_{A_n}-\chi_{A_n}\|_p}{h(A_n)^{\frac{1}{p}}} = 0$ and the assumptions it follows $\lim_{n\to\infty} \frac{\|T_{k+1}\chi_{A_n}-\chi_{A_n}\|_p^p}{h(A_n)} = 0$. □

For the canonical sequence $(S_n)_{n\in\mathbb{N}}$ we find a close connection to property (H), similar to Theorem 3.10.

Theorem 3.19. *(a) If condition (H) is satisfied, then $(S_n)_{n\in\mathbb{N}_0}$ satisfies property (F_1).*

(b) $(S_n)_{n\in\mathbb{N}_0}$ satisfies (F_p) either for all $1 \leq p < \infty$ or for none.

(c) If there exists $c > 0, n_0 \in \mathbb{N}_0$ such that $|a_n| \geq c$ or $|c_n| \geq c$ for all $n \geq n_0$ and if $(S_n)_{n\in\mathbb{N}_0}$ satisfies (F_1) then the hypergroup fulfils property (H).

Proof. (a): Clearly (i)-(iii) are satisfied. By Lemma 3.18 it suffices to consider $k = 1$. From $\chi_{S_n}(l) - T_1\chi_{S_n}(l) = 0$ for $l = 0, \ldots, n-1$ and $l = n+2, n+3, \ldots$ and $\chi_{S_n}(n) - T_1\chi_{S_n}(n) = 1 - (b_n + c_n) = a_n$, $\chi_{S_n}(n+1) - T_1\chi_{S_n}(n+1) = -c_{n+1}$ we obtain $\|\chi_{S_n} - T_1\chi_{S_n}\|_1 = a_n h(n) + c_{n+1}h(n+1) = 2a_n h(n)$. By (H) and $0 < a_n < 1$ (F_1) follows.
(b): Let $(S_n)_{n\in\mathbb{N}_0}$ satisfy (F_1). By Lemma 3.18, it is sufficient to show (iv) for (F_p) only for $k = 1$.
First we notice that

$$\begin{aligned}
\|T_1\chi_{A_n} - \chi_{A_n}\|_p^p &= a_n{}^p h(n) + c_{n+1}{}^p h(n+1) = a_n{}^p h(n) + a_n h(n) c_{n+1}{}^{p-1} \\
&= h(n)(a_n{}^p + a_n c_{n+1}{}^{p-1})
\end{aligned}$$

and because of $0 < a_n, c_n < 1$ for all $n \in \mathbb{N}$ we can see that

$$\|T_1\chi_{A_n} - \chi_{A_n}\|_p^p \leq \|T_1\chi_{A_n} - \chi_{A_n}\|_1$$

for $1 \leq p \leq q$.
The other direction has recently been shown by Lasser and Skantharajah in [21].

(c):$(S_n)_{n\in\mathbb{N}}$ satisfying (F_1) can be written as

$$0 = \lim_{n\to\infty} \frac{\|\chi_{S_n} - T_1\chi_{S_n}\|_1}{\sum_{k=0}^{n} h(k)} = \lim_{n\to\infty} \frac{a_n h(n) + c_{n+1}h(n+1)}{\sum_{k=0}^{n} h(k)}$$

and with at least one of the sequences $(a_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}}$ being bounded away from zero (H) follows. □

For summing sequences Proposition 3.12 gives a condition when there doesn't exist one. For property (F_1) we can give a similar result:

Proposition 3.20. *If* $\frac{h(n)}{\sum_{k=0}^{n} h(k)} \geq M_1 > 0$ *and* $c_n \geq M_2 > 0$ *for all* $n \in \mathbb{N}_0$ *then there does not exist a sequence* $(A_n)_{n\in\mathbb{N}}$ *satisfying property* (F_1).

Proof. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets satisfying (i)-(iii) of Definition 3.16 define $m_n := \max A_n$. Then

$$\frac{1}{h(A_n)}\|T_1\chi_{A_n} - \chi_{A_n}\|_1 \geq \frac{1}{h(A_n)} c_{m_n+1} h(m_n+1) \geq M_1 M_2 > 0.$$

□

To illustrate the results of Theorem 3.10 and Theorem 3.19 we present some examples.

(i) Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ with $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$ induce a polynomial hypergroup on $\mathbb{N}_0$. The orthogonalization measure on $[-1, 1]$ is $d\pi(x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta dx$. The explicit form of the recurrence coefficients a_n, b_n, c_n may be found in [4] or [18]. The Haar weights

$$h(n) = \frac{(2n+\alpha+\beta+1)(\alpha+\beta+1)_n(\alpha+1)_n}{(\alpha+\beta+1)n!(\beta+1)_n},$$

are of polynomial growth, more precisely $h(n) = O(n^{2\alpha+1})$ as $n \to \infty$. Hence we conclude that $(S_n)_{n\in\mathbb{N}_0}$ is a summing sequence and satisfies (F_p) for every $p \in [1, \infty[$.

(ii) Little q-Legendre polynomials $R_n^{(q)}(x)$ with $0 < q < 1$ define a polynomial hypergroup on $\mathbb{N}_0$, see [17] or [10]. The recurrence coefficients are

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}$$

$$b_n = \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})}$$

$$c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}$$

for $n \in \mathbb{N}$, with starting values $a_0 = \frac{1}{q+1}$ and $b_0 = \frac{q}{q+1}$. The Haar weights satisfy

$$\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = \frac{1}{q} > 1.$$

Hence $h(n)$ is of exponential growth. Moreover, $a_n \to 0, c_n \to 0, b_n \to 1$ and $\frac{a_n}{c_{n+1}} \to \frac{1}{q}$. Hence we conclude that $(S_n)_{n\in\mathbb{N}}$ is not a summing sequence. Nevertheless $(S_n)_{n\in\mathbb{N}}$ satisfies property (F_p) for each $1 \leq p < \infty$.

(iii) Orthogonal polynomials connected with homogeneous trees $R_n(x; a)$ with $a \geq 2$ are determined by the recurrence coefficients

$$a_n = \frac{a-1}{a}, b_n = 0, c_n = \frac{1}{a}, n \in \mathbb{N}$$

and $a_0 = 1, b_0 = 0$, see [18] The Haar weights are $h(0) = 1, h(n) = a(a-1)^{n-1}, n \in \mathbb{N}$. From the preceding results we can derive that $(S_n)_{n\in\mathbb{N}}$ is not a summing sequence and it does not satisfy (F_p) for any $1 \leq p < \infty$.

There are examples where the condition "$|a_n| \geq c$ for almost all $n \in \mathbb{N}_0$" of Theorem 3.19(c) is satisfied, as in the hypergroups generated by the ultraspherical polynomials.

For polynomial hypergroups where the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ satisfies property (F_1) instead of the stronger assumption of being a summing sequence we can still prove a representation of the unique translation invariant mean on $\mathcal{AC}(\mathbb{N}_0)$ via the same limit as in Theorem 3.14. To that end, we will need the following proposition:

Proposition 3.21. *Let the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ satisfy (F_1) and let $g_n : l^\infty(\mathbb{N}_0) \to \mathbb{C}, g_n(\varphi) := \frac{1}{h(S_n)} \sum_{k\in S_n} \varphi(k)h(k)$. Then*

$$\lim_{n\to\infty} |g_n(T_1\varphi) - g_n(\varphi)| = 0 \textit{ for all } \varphi \in l^\infty(\mathbb{N}_0)$$

Proof. We have

$$\begin{aligned}
g_n(T_1\varphi) &= \frac{1}{h(S_n)} \sum_{k=0}^{n} \sum_{j=|k-1|}^{k+1} g(1,k;j)\varphi(j)h(k) \\
&= \frac{1}{h(S_n)} \sum_{k=0}^{n} \sum_{j=|k-1|}^{k+1} g(1,j;k)\varphi(j)h(j) \\
&= \frac{1}{h(S_n)} \sum_{k=0}^{n} \big(a_{|k-1|}\varphi(|k-1|)h(|k-1|) + b_k\varphi(k)h(k) + c_{k+1}\varphi(k+1)h(k+1)\big) \\
&= g_n(\varphi) + \frac{1}{h(S_n)} \left(a_1\varphi(1)h(1) - a_n\varphi(n)h(n) + c_{n+1}\varphi(n+1)h(n+1)\right) \\
&= g_n(\varphi) + \frac{1}{h(S_n)} \left(a_1\varphi(1)h(1) + a_n h(n)(\varphi(n+1) - \varphi(n))\right)
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} |g_n(T_1\varphi) - g_n(\varphi)| \leq 2\|\varphi\|_\infty \lim_{n\to\infty} \frac{a_n h(n)}{h(S_n)} = \|\varphi\|_\infty \lim_{n\to\infty} \frac{\|\chi_{S_n} - T_1\chi_{S_n}\|_1}{h(S_n)} = 0$$

□

Theorem 3.22. *Let the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ satisfy (F_1) and let $\varphi \in \mathcal{AC}(\mathbb{N}_0)$. Then the sequence $(\frac{1}{h(S_n)} \sum_{k\in S_n} \varphi(k)h(k))_{n\in\mathbb{N}_0}$ converges and the unique translation invariant mean on $\mathcal{AC}(\mathbb{N}_0)$ is given by*

$$M(\varphi) = \lim_{n\to\infty} \frac{1}{h(S_n)} \sum_{k\in S_n} \varphi(k)h(k).$$

Proof. It is sufficient to prove the representation for real-valued functions $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$. For such φ, let $\mathcal{P}(\varphi) := \limsup_{n\to\infty} g_n(\varphi)$, where g_n are defined as in Proposition 3.21. $\mathcal{P}$ satisfies $\mathcal{P}(\varphi+\psi) \leq \mathcal{P}(\varphi)+\mathcal{P}(\psi)$ and $\mathcal{P}(\alpha\varphi) = \alpha\mathcal{P}(\varphi)$ for $\alpha \geq 0$. By the Hahn-Banach theorem there exists a linear functional M_0 on $\mathcal{AC}_r(\mathbb{N}_0)$ such that

$$-\mathcal{P}(-\varphi) \leq M_0(\varphi) \leq \mathcal{P}(\varphi) \text{ for all } \varphi \in \mathcal{AC}_r(\mathbb{N}_0).$$

With Proposition 3.21 one can see that $\mathcal{P}(T_1\varphi - \varphi) = -\mathcal{P}(-T_1\varphi + \varphi) = 0$ for all $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$. This shows that $M_0(T_1\varphi) = M_0(\varphi)$. Also $M_0(\varphi) \geq 0$ when $\varphi \geq 0$, $M_0(\mathbf{1}) = 1$ and $\|M_0\| = 1$ can easily be seen with the inequality $-\mathcal{P}(-\varphi) \leq M_0(\varphi) \leq \mathcal{P}(\varphi)$. By Proposition 3.3 M_0 is a translation invariant mean. Thus by the uniqueness of the translation invariant mean on $\mathcal{AC}(\mathbb{N}_0)$ we get $M_0(\varphi) = M(\varphi)$ for all $\varphi \in \mathcal{AC}(\mathbb{N}_0)$. Assume there is a function $\varphi \in \mathcal{AC}_r(\mathbb{N}_0)$ such that

$$-\mathcal{P}(-\varphi) = \liminf_{n\to\infty} g_n(\varphi) < \limsup_{n\to\infty} g_n(\varphi) = \mathcal{P}(\varphi).$$

Then by the Hahn-Banach theorem there would be two functionals M_0 and M_1 which would be equal to M on $\mathcal{AC}_r(\mathbb{N}_0)$, a contradiction to the assumed uniqueness of the mean. Thus the limit

$$\lim_{n\to\infty} \frac{1}{h(S_n)} \sum_{k\in S_n} \varphi(k)h(k)$$

exists and is equal to $M(\varphi)$. □

With the results of this and the previous section it makes sense to introduce the following class of functions:

Definition 3.23. *Let $(S_n)_{n\in\mathbb{N}_0}$ be the canonical sequence. Define*

$$\mathcal{L} := \{\varphi \in l^\infty(\mathbb{N}_0) : \lim_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} \varphi(k)h(k) \text{ exists}\}$$

Using this notation, Theorem 3.22 states that if $(S_n)_{n\in\mathbb{N}_0}$ satisfies (F_1), then $\mathcal{AC}(\mathbb{N}_0) \subseteq \mathcal{L}$. For a start, we can give some properties of $\mathcal{L}$.

Proposition 3.24. *$\mathcal{L}$ is a translation invariant closed linear subspace of $l^\infty(\mathbb{N}_0)$ with $c_0 \subseteq \mathcal{L}$.*

Proof. It is easy to see that $\mathcal{L}$ is a closed linear subspace of $l^\infty(\mathbb{N}_0)$ with $c_0 \subseteq \mathcal{L}$. By Proposition 3.21 we get $T_1\varphi \in \mathcal{L}$. But as $\mathcal{L}$ is a linear space and by Lemma 3.4 every T_m can be written as a convex combination of powers of T_1, the proof is finished. □

Since $c_0 \subseteq \mathcal{AC}(\mathbb{N}_0)$ the previous Proposition states a new inclusion only for the case when $\mathcal{AC}(\mathbb{N}_0) \not\subseteq \mathcal{L}$, i.e. when there are no summing sequences and when $(S_n)_{n\in\mathbb{N}_0}$ does not satisfy (F_1).

The inclusion $\mathcal{L} \subseteq \mathcal{AC}(\mathbb{N}_0)$ is not true in general, i.e. the existence of the limit $\lim\limits_{n\to\infty} \frac{1}{h(S_n)} \sum\limits_{k=0}^{n} \varphi(k)h(k)$ for a $\varphi \in l^\infty(\mathbb{N}_0)$ is in general not sufficient for the uniqueness of $m(\varphi)$. For example, even if condition (H) holds, i.e. the inclusion $\mathcal{AC}(\mathbb{N}_0) \subseteq \mathcal{L}$ is valid, we can give an example that these to sets do not always coincide:

Example 3.25. *Let the hypergroup structure on $\mathbb{N}_0$ be induced by the Chebyshev polynomials of the first kind (see section 1.2). Define a set $A \subseteq \mathbb{N}_0$ by*

$$\chi_A(n) = \begin{cases} 1 & \text{for } n = 2^m, 2^m \pm 1, \ldots, 2^m \pm m \text{ and } m \text{ even} \\ 0 & \text{else} \end{cases}$$

In the proof of Corollary 4.3, where this set is also used, we can see that $\chi_A \notin \mathcal{AC}(\mathbb{N}_0)$. So it remains to be shown that $\chi_A \in \mathcal{L}$. For $n \geq 2$ there exists a unique $m \in \mathbb{N}$ such that $2^m + m \geq n > 2^{m-1} + (m-1)$. With that we get

$$\begin{aligned} \sum_{k=0}^{n} \chi_A(k)h(k) &\leq \sum_{k=0}^{2^m+m} \chi_A(k)h(k) \leq \sum_{j=0}^{m} \sum_{i=2^j-j}^{2^j+j} \chi_A(i)h(i) \\ &\leq 2\sum_{j=0}^{m} 2j+1 = 2m^2+4m+2 \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{H(S_n)} \sum_{k=0}^{n} \chi_A(k)h(k) &\leq \frac{1}{H(S_{2^{m-1}+(m-1)})} \left(2m^2+4m+2\right) \\ &\leq \frac{1}{2^m+2m-1} \left(2m^2+4m+2\right) \to 0 \text{ as } m \to \infty \end{aligned}$$

This example also shows that it is difficult to find bounds for $m(\varphi)$ other than

$$\liminf_{n\to\infty} \varphi(n) \leq m(\varphi) \leq \limsup_{n\to\infty} \varphi(n)$$

that hold for all translation invariant means m and all $\varphi \in l^\infty(\mathbb{N}_0)$. Especially

$$\limsup_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} \varphi(k)h(k) \ngeq m(\varphi)$$

has been shown as there exists a mean m such that $m(\chi_A) = 1$ and the left-hand side was 0 for this sequence. Another possible bound that is not as strict is

$$\sup_{l\in\mathbb{N}_0} \limsup_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} T_l\varphi(k)h(k)$$

We will use the same function as in Example 3.25 to show that this inequality is also not true in general.

Example 3.26. *Let the hypergroup structure on $\mathbb{N}_0$ be induced by the Chebyshev polynomials of the first kind (see section 1.2). Define a set $A \subseteq \mathbb{N}_0$ by*

$$\chi_A(n) = \begin{cases} 1 & \text{for } n = 2^m, 2^m \pm 1, \ldots, 2^m \pm m \text{ and } m \text{ even} \\ 0 & \text{else} \end{cases}$$

In the proof of Corollary 4.3 we show that there exists a translation invariant mean m such that $m(\chi_A) = 1$. The term in question for the upper bound on the other hand gives

$$\begin{aligned}
&\sup_{l\in\mathbb{N}_0} \limsup_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} T_l\varphi(k)h(k) = \\
&= \sup_{l\in\mathbb{N}_0} \limsup_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} \left(\frac{1}{2}\chi_A(|k-l|) + \frac{1}{2}\chi_A(k+l) \right) h(k) \\
&\leq \sup_{l\in\mathbb{N}_0} \limsup_{n\to\infty} \frac{1}{h(S_n)} \left(\frac{1}{2}\sum_{k=0}^{l} \chi_A(k)h(k) + \frac{1}{2}\sum_{k=0}^{n} \chi_A(k)h(k) + \frac{1}{2}\sum_{k=0}^{n+l} \chi_A(k)h(k) \right)
\end{aligned}$$

For $n \geq 2$ there exists a unique $N \in \mathbb{N}$ such that $2^N + N \geq n > 2^{N-1} + (N-1)$. Assuming $n \geq l$ we get

$$\frac{1}{h(S_n)} \left(\frac{1}{2}\sum_{k=0}^{l} \chi_A(k)h(k) + \frac{1}{2}\sum_{k=0}^{n} \chi_A(k)h(k) \right) \quad \leq \quad \frac{1}{h(S_n)} \sum_{k=0}^{n} \chi_A(k)h(k) \to 0$$

as in Example 3.25. For the third term, with n, l and N fixed, choose ΔN minimally such that

$$2^{N+\Delta N} + (N + \Delta N) \geq n + l.$$

Then

$$\frac{1}{h(S_n)} \sum_{k=0}^{n+l} \chi_A(k)h(k) \leq \frac{2(N+\Delta N)^2 + 4(N+\Delta N) + 2}{2^N + 2N - 1}$$

again as in example 3.25. Since ΔN, which is dependent on n and l, is decreasing as $n \to \infty$, there is a maximal ΔN and so the third term also tends to zero and in the end

$$\sup_{l\in\mathbb{N}_0} \limsup_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} T_l\chi_A(k)h(k) = 0.$$

4 Non-Uniqueness of translation invariant means

In this chapter we will prove the existence of two different means on l^∞ for polynomial hypergroups. The proof uses an idea first proposed by Rudin[25] for invariant means on groups. But due to zeroes of functions not being shifted by the hypergroup translation, the proof had to be modified accordingly.

For the first part of this chapter, let K be a commutative hypergroup.

Definition 4.1. *A set $E \subseteq K$ is called* **permanently positive** *(in short* **PP***) if*

$$h\left(\bigcap_{i=1}^{n} T_{x_i} E\right) > 0$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in K$.
$\Omega \subseteq \mathcal{P}(K)$ is called **PP-filter** *if*

(i) *every $A \in \Omega$ is PP*

(ii) *$A, B \in \Omega \Rightarrow A \cap B \in \Omega$*

For $f \in L^\infty(K)$, let $Z(f)$ denote the set of all zeroes of f in the sense that x is a zero if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_{B_\delta(x)} f(y)dm(y) < \varepsilon$.

Theorem 4.2. *Let Ω be a PP-Filter in K. Let $J := \{f \in L^\infty(K) : Z(f)$ contains some member of $\Omega\}$ and $I \subseteq J$ such that $T_x f \in J$ for all $x \in K$ whenever $f \in I$. Then there exists a translation invariant mean M on $L^\infty(K)$ such that $Mf = 0$ for all $f \in I$.*

Proof. $L^\infty(K)$ is a Banach algebra with respect to pointwise multiplication. Because of (ii) in Definition 4.1, J is an ideal. Since $\mathbf{1} \notin J$, J is contained in a maximal ideal $J_{\max}$ and as $J_{\max}$ is maximal, the quotient ring $L^\infty(K)/J_{\max}$ is a field. By the Gelfand-Mazur theorem that field is isomorphic to $\mathbb{C}$, and there is a homomorphism $h : L^\infty(K) \to \mathbb{C}$ with $\ker(h) = J_{\max}$.
Since K is amenable, there exists a translation invariant mean Λ on $L^\infty(K)$. Since $\|h\| = 1$, the formula

$$\Phi f(x) = h(T_x f)$$

associates with each $f \in L^\infty(K)$ a function $\Phi f \in L^\infty(K)$, so one can define

$$Mf := \Lambda \Phi f$$

for $f \in L^\infty(K)$.
By the definition of I, $T_x f \in J$ whenever $f \in I \subseteq J$, and so $Mf = 0$ for every $f \in I$.
$M\mathbf{1} = 1$ as $h(\mathbf{1}) = 1$ and $\Lambda\mathbf{1} = 1$.
Since $\|h\| = 1$, we have $|Mf| \leq \|f\|_\infty$. So only the translation invariance of M remains to be shown. We can see that

$$\begin{aligned}\Phi(T_x f)(y) &= h(T_y T_x f) = h\left(\int_K T_z f d\omega(x,y)(z)\right) \overset{(*)}{=} \int_K h(T_z f) d\omega(x,y)(z) \\ &= \int_K \Phi f(z) d\omega(x,y)(z) = T_x \Phi f(y)\end{aligned}$$

where $(*)$ holds because h is continuous as a homomorphism from a Banach algebra in $\mathbb{C}$. With that we can conclude that

$$M(T_x f) = \Lambda\Phi T_x f = \Lambda T_x \Phi f = \Lambda\Phi f = Mf$$

and so M is a translation invariant mean with the required property. □

This general theorem is the key to show that on polynomial hypergroups there cannot exist only one mean. The idea is to construct a set A such that A and its translates as well as the complement A^C along with its translates lie in two suitably defined PP-filters. Then Theorem 4.2 guarantees the existence of means M_1 and M_2 such that $M_1(\chi_{A^C}) = 0$ and $M_2(\chi_A) = 0$ but as $\chi_A + \chi_{A^C} = \mathbf{1}$ and $M_1(\mathbf{1}) = 1$ that completes the proof.

Corollary 4.3. *For polynomial hypergroups there exists more than one mean on l^∞.*

Proof. For $A \subseteq \mathbb{N}_0$ define

$$L_A(n) := \begin{cases} \max\limits_{N \in \{0,\dots,2^n\}} \left\{ \prod\limits_{k=-N}^{N} \chi_A(2^n - k) \cdot \left(\sum\limits_{k=-N}^{N} \chi_A(2^n - k) \right) \right\} & n \text{ even} \\ \max\limits_{N \in \{0,\dots,2^n\}} \left\{ \prod\limits_{k=-N}^{N} (1 - \chi_A(2^n - k)) \cdot \left(\sum\limits_{k=-N}^{N} (1 - \chi_A(2^n - k)) \right) \right\} & n \text{ odd} \end{cases}$$

$$\widetilde{L}_A(n) := \begin{cases} \max\limits_{N \in \{0,\dots,2^n\}} \left\{ \prod\limits_{k=-N}^{N} (1 - \chi_A(2^n - k)) \cdot \left(\sum\limits_{k=-N}^{N} (1 - \chi_A(2^n - k)) \right) \right\} & n \text{ even} \\ \max\limits_{N \in \{0,\dots,2^n\}} \left\{ \prod\limits_{k=-N}^{N} \chi_A(2^n - k) \cdot \left(\sum\limits_{k=-N}^{N} \chi_A(2^n - k) \right) \right\} & n \text{ odd} \end{cases}$$

$L_A(n)$ counts the number of contiguous elements of A symmetrically around 2^n for n even and contiguous numbers in $\mathbb{N}_0 \setminus A$ symmetrically around 2^n for n odd, whereas $\widetilde{L}_A(n)$ does the opposite.
Let $\Omega := \{A \subseteq \mathbb{N}_0 : \liminf\limits_{n\to\infty} \frac{L_A(n)}{n} > 0\}$ and $\widetilde{\Omega} := \{A \subseteq \mathbb{N}_0 : \liminf\limits_{n\to\infty} \frac{\widetilde{L}_A(n)}{n} > 0\}$. With this definition Ω and $\widetilde{\Omega}$ are PP-filters, as every $A \in \Omega(\widetilde{\Omega})$ is permanently positive and

with $A, B \in \Omega(\widetilde{\Omega})$ we have $A \cap B \in \Omega(\widetilde{\Omega})$ and $T_m A \in \Omega(\widetilde{\Omega})$ for all $m \in \mathbb{N}_0$.
Now define a set $A \subseteq \mathbb{N}_0$ such that

$$\chi_A(n) = \begin{cases} 1 & \text{for } n = 2^m, 2^m \pm 1, \ldots, 2^m \pm m \text{ and } m \text{ even} \\ 0 & \text{else} \end{cases}$$

A lies in Ω and so χ_{A^C} has zeroes on an element of a PP-filter, as required in the previous theorem. For every $k \in \mathbb{N}_0$, there exists a set $B \in \Omega$ such that $B \subseteq Z(T_k \chi_{A^C})$. So by Theorem 4.2, there exists a mean M_1 on l^∞ such that $M_1(\chi_{A^C}) = 0$. By the same argument there exists a mean M_2 such that $M_2(\chi_A) = 0$. But since $1 = M_2(\mathbf{1}) = M_2(\chi_A) + M_2(\chi_{A^C}) = M_2(\chi_{A^C})$, we have shown the existence of two different means. □

This result proves in general that on any polynomial hypergroup there exist two different translation invariant means. However, as we have seen in the previous chapter, for some functions $\varphi \in l^\infty$ the mean is uniquely determined. To get a criterion for when $\varphi \notin \mathcal{AC}(\mathbb{N}_0)$ we can reverse Theorem 3.14 and Theorem 3.22:

Corollary 4.4. *Let $(A_n)_{n\in\mathbb{N}_0}$ be a summing sequence or let $(A_n)_{n\in\mathbb{N}_0} = (S_n)_{n\in\mathbb{N}_0}$ satisfy (F_1), and let $\varphi \in l^\infty$. If*

$$\lim_{n\to\infty} \frac{1}{h(A_n)} \sum_{k\in A_n} \varphi(k)h(k)$$

does not exist, then $\varphi \notin \mathcal{AC}(\mathbb{N}_0)$.

Proof. Apply Theorem 3.14 or Theorem 3.22. □

On the other hand, if this limit does exist for a $\varphi \in l^\infty$, one can at least assert that it is the representation of a mean $m(\varphi)$:

Corollary 4.5. *Let $(A_n)_{n\in\mathbb{N}_0}$ be a summing sequence or let $(A_n)_{n\in\mathbb{N}_0} = (S_n)_{n\in\mathbb{N}_0}$ satisfy (F_1), and let $\varphi \in l^\infty$. If*

$$\lim_{n\to\infty} \frac{1}{h(A_n)} \sum_{k\in A_n} \varphi(k)h(k) =: c,$$

then there is a translation invariant mean $m \in (l^\infty)^*$ such that $m(\varphi) = c$.

Proof. This follows from the first half of the proofs of Theorems 3.14 and 3.22. □

In the case of $\varphi \in \mathcal{AC}(\mathbb{N}_0)$, i.e. $m(\varphi)$ is uniquely determined by φ, the limit automatically existed and its value was fixed. In the general case we have to assume the existence of the limit and can only assert that there exists a mean which is represented by that limit, and the mean is not fixed.

So if $\varphi \notin \mathcal{AC}(\mathbb{N}_0)$ there exists more than one mean, and we can even find an example where there exist two summing sequences that represent different means in the sense of Corollary 4.5. The set A is the one used in the proof of Corollary 4.3.

Example 4.6. *Let the hypergroup structure on $\mathbb{N}_0$ be induced by the Chebyshev polynomials of the first kind (see section 1.2). Define a set $A \subseteq \mathbb{N}_0$ by*

$$\chi_A(n) = \begin{cases} 1 & \text{for } n = 2^m, 2^m \pm 1, \ldots, 2^m \pm m \text{ and } m \text{ even} \\ 0 & \text{else} \end{cases}$$

For the canonical sequence $(S_n)_{n\in\mathbb{N}_0}$ we have already seen in Example 3.25 that

$$\lim_{n\to\infty} \frac{1}{h(S_n)} \sum_{k=0}^{n} \chi_A(k)h(k) = 0$$

and this represents one of the means found in the proof of Corollary 4.3.
In the second half of this example we will construct a sequence of sets $(A_n)_{n\in\mathbb{N}_0}$, show that it is a summing sequence and then show that the corresponding limit is 1. *To that end, let $B_k := \{2^{2k}, 2^{2k} \pm 1, \ldots, 2^{2k} \pm k\}$ for $k \in \mathbb{N}_0$ and define*

$$A_0 := B_0, A_{n+1} := A_n \cup B_{n+1} \cup \{\underbrace{\min\{k \in \mathbb{N}_0 : k \notin (A_n \cup B_{n+1})\}}_{=:c_{n+1}}\}.$$

One can see that $A_n = \bigcup\limits_{l=0}^{n} B_l \cup \{c_1, \ldots, c_n\}$ as a disjoint union (by the definition of c_i) and so $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}_0$, $\bigcup\limits_{n=0}^{\infty} A_n = \mathbb{N}_0$ and $h(A_n) < \infty$ for all $n \in \mathbb{N}_0$ as the sets are finite, i.e. (i)-(iii) of Definition 3.9 are satisfied. For (iv) we will look at the symmetric differences $T_kA_n\Delta A_n$. Let k be given. Then we get

$$T_kA_n \subseteq \bigcup_{l=0}^{n} \{2^{2l}, 2^{2l} \pm 1, \ldots, |2^{2l} \pm (2l+k)|\} \cup \{0, 1, \ldots, c_n + k\}$$

not necessarily disjoint anymore, and thus

$$T_kA_n \setminus A_n \subseteq \bigcup_{l=0}^{n} \{|2^{2l} \pm (2l+1)|, \ldots, |2^{2l} \pm (2l+k)|\} \cup \{c_n + 1, \ldots, s_n + k\}.$$

For the Haar measure of this set we get

$$h(T_kA_n \setminus A_n) \leq \sum_{l=0}^{n} (4k) + 2k = 4k \cdot n + 6k.$$

On the other hand, for $l \geq k$ we find $B_l \subseteq T_kB_l$ and $\{0, \ldots, c_n\} \subseteq \{0, \ldots, c_n + k\}$ and so $A_n \setminus T_kA_n \subseteq \bigcup\limits_{l=0}^{k-1} B_l$. Applying the Haar measure this leads to

$$h(A_n \setminus T_kA_n) \leq \sum_{l=0}^{k-1} (4l+2) = 2k(k-1) + 2k = 2k^2$$

and combined this yields

$$h(T_k A_n \Delta A_n) \leq 4k \cdot n + 2k^2 + 6k.$$

To check (iv) of Definition 3.9 we need to compare this to $h(A_n)$ *which is given by*

$$h(A_n) = \sum_{l=0}^{n} h(B_l) + h(\{\underbrace{c_1}_{=0}, \ldots, c_n\}) = \sum_{l=0}^{n} (8l+2) + 2n - 1 = 4n^2 + 8n + 1$$

and thus

$$\frac{h(T_k A_n \Delta A_n)}{h(A_n)} \leq \frac{4k \cdot n + 2k^2 + 6k}{4n^2 + 8n + 1} \to 0 \text{ for every } k \in \mathbb{N}_0 \text{ as } n \to \infty.$$

This concludes the proof that $(A_n)_{n \in \mathbb{N}_0}$ *is a summing sequence and now we want to check whether* $\lim\limits_{n\to\infty} \frac{1}{h(A_n)} \sum\limits_{k \in A_n} \chi_A(k) h(k)$ *exists. With*

$$\begin{aligned} \sum_{k \in A_n} \chi_A(k) &= \sum_{l=0}^{n} \sum_{k \in B_l} \underbrace{\chi_A(k)}_{=1} h(k) + \sum_{k=1}^{n} \underbrace{\chi_A(c_k)}_{=0} h(c_k) \\ &= \sum_{l=0}^{n} \sum_{k \in B_l} 2 = 2 \sum_{l=0}^{n} (4l+1) = 4n^2 + 6n + 2 \end{aligned}$$

we get

$$\frac{1}{h(A_n)} \sum_{k \in A_n} \chi_A(k) h(k) = \frac{4n^2 + 6n + 2}{4n^2 + 8n + 1} \to 1$$

which represents $M_1(\chi_A)$ *from the proof of Corollary 4.3.*

In this example the canonical sequence $(S_n)_{n \in \mathbb{N}_0}$ represented $M_2(\chi_A) = 0$ and a custom-made sequence $(A_n)_{n \in \mathbb{N}_0}$ allowed for $M_1(\chi_A) = 1$. As the calculations in the proof show, even in the simple case of the hypergroup induced by the Chebyshev polynomials of the first kind, it is no easy task to find a summing sequence that represents a mean one knows beforehand. If in contrast one has a summing sequence such that the limit for a given function $\varphi \in l^\infty(\mathbb{N}_0)$ exists one can in light of Corollary 4.5 just define the mean to be its limit.

List of Symbols

a_n	coefficients of the three-term recurrence relation
$(A_n)_{n\in\mathbb{N}_0}$	summing sequence
$\mathcal{AC}(K)$	almost convergent functions on K, see Definition 2.4
b_n	coefficients of the three-term recurrence relation
$\mathcal{C}$	continuous functions
$\mathcal{C}_C$	continuous functions with compact support
c_n	coefficients of the three-term recurrence relation
coA	convex hull of A
χ	hypergroup character
χ_A	characteristic function of the set A
ε_x	Dirac measure at the point x
$g(m,n;k)$	linearization coefficients of a sequence of orthogonal polynomials
h	Haar measure on a polynomial hypergroup
$(K_n)_{n\in\mathbb{N}_0}$	Følner sequence
l^∞	$\{f:\mathbb{N}_0\to\mathbb{C}:\sup\limits_{n\in\mathbb{N}_0}\lvert f(n)\rvert h(n)<\infty\}$
λ	Haar measure on a locally compact Abelian group G
m	translation invariant mean
$\mathfrak{M}(K)$	set of all means on K
$\mathfrak{M}_t(K)$	set of translation invariant means on K
μ	Haar measure on a commutative hypergroup K
$O(f(n))$	$g(n)=O(f(n))$ if $\exists n_0\in\mathbb{N}, M\in\mathbb{R}:\lvert g(n)\rvert\le M\lvert f(n)\rvert\forall n\ge n_0$
$\omega(x,y)$	hypergroup convolution, see Definition 1.1
$P_{00}(K)$	probability measures on K with compact support
$\mathcal{P}(X)$	power set of X
π	orthogonalization measure of a sequence of orthogonal polynomials
R_n	polynomial from a sequence of orthogonal polynomials $(R_n)_{n\in\mathbb{N}_0}$
$(S_n)_{n\in\mathbb{N}_0}$	canonical summing sequence, $S_n:=\{0,1,\dots,n\}$
T_x	translation operator
$\mathcal{WAP}(K)$	weakly almost periodic functions, see Definition 2.5
$\mathbf{1}$	constant function $f\equiv 1$
$\overline{A}$	closure of the set A
$\overline{A}^\infty$	closure of the set A in the uniform topology
$\overline{f}$	complex conjugation

Bibliography

[1] R. Askey, Linearization of the product of orthogonal polynomials, in R. Gunning, Problems in Analysis, Princeton University Press, pp. 223-228, 1970.

[2] S. Banach, Sur le problème de mésure, Fund. Math., vol. 4, pp. 7-33, 1923.

[3] S. Banach and A. Tarski, Sur la décomposition des ensembles de points en parts respectivement congruents, Fund. Math., vol. 6, pp. 244-277, 1924.

[4] W.R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, de Gruyter, Berlin, 1994.

[5] H. Bohr, Zur Theorie der fastperiodischen Funktionen I, Acta Math., vol. 45, pp. 29-127, 1925.

[6] R.B. Burckel, Weakly Almost Periodic Functions on Semigroups, Gordon and Beach, New York, 1970.

[7] J. Dixmier, Les moyennes invariantes dans les semi-groupes et leurs applications, Acta Sci. Math. Szeged 12A, pp. 213-227, 1950.

[8] N. Dunford and J.T. Schwartz, Linear Operators I, Wiley&Sons, 1988.

[9] F. Filbir, Approximationsprozesse auf kommutativen Hypergruppen, Shaker Verlag, Aachen, 2000.

[10] F. Filbir, R. Lasser and R. Szwarc, Reiter's Condition P_1 and Approximate Identities for Polynomial Hypergroups, Monatsh. Math., vol. 143, pp. 189-203, 2004.

[11] E. Følner, On groups with full Banach mean value, Mathematica Scandinavica, vol. 3, pp. 243-254, 1955.

[12] F. Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914.

[13] E.H. Hewitt and K.A. Ross, Abstract Harmonic Analysis I, Springer-Verlag, Berlin, 1963.

[14] V. Hösel, On the estimation of covariance functions of Pn-weakly stationary processes, Stoch. Anal., vol. 16, no. 4, pp. 607-629, 1998.

[15] V. Hösel, M. Hofmann and R. Lasser, Means and Følner Condition on Polynomial Hypergroups, Mediterranean Journal of Mathematics, vol. 7, pp. 75-88, 2010.

[16] V. Hösel and R. Lasser, Prediction of weakly stationary sequences on polynomial hypergroups, Annals of Probability, vol. 31, no. 1, pp. 93-114, 2003.

[17] T.H. Koornwinder, The addition formula for little q-Legendre polynomials and the SU(2) quantum group, SIAM J. Math. Anal., vol. 22, no. 1, pp. 295-301, 1991.

[18] R. Lasser, Orthogonal Polynomials and Hypergroups, Rend. Mat., vol. 3, pp. 185-209, 1983.

[19] R. Lasser, Orthogonal Polynomials and Hypergroups II – The Symmetric Case, Trans. AMS, vol. 341, no. 2, pp. 749-770, 1994.

[20] R. Lasser, On α-amenability of commutative hypergroups, Infinite Dimensional Harmonic Analysis 4, Eds. J. Hilgert et al., World Scientific, Tokyo, pp. 184-195, 2008.

[21] R. Lasser and M. Skantharajah, Reiter's Condition for Amenable Hypergroups, Monatshefte Math., vol. 163, pp. 327-338, 2011.

[22] A.T. Lau, Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups, Fund. Math., vol.118, pp. 161-175, 1983.

[23] H. Lebesgue, Leçons sur l'intégration et la recherche des fonctions primitives, Gauthier-Villars, Paris, 1904.

[24] A.L.T. Paterson, Amenability, Mathematical surveys and monographs, vol. 29, American Mathematical Society, Providence, Rhode Island, 1988.

[25] W. Rudin, Invariant means on L^∞, Studia Mathematica, T. XLIV, pp. 219-227, 1972.

[26] M. Skantharajah, Amenable Hypergroups, Illinois Journal of mathematics, vol. 36, no. 1, pp. 15-46, 1992.

[27] R. Szwarc, Orthogonal polynomials and a discrete boundary value problem I, SIAM Journal on Mathematical Analysis, vol. 23, pp. 959-964, 1992.

[28] R. Szwarc, Orthogonal polynomials and a discrete boundary value problem II, SIAM Journal on Mathematical Analysis, vol. 23, pp. 965-969, 1992.

[29] A. Tarski, Algebraische Fassung des Massproblems, Fund. Math., vol. 31, pp. 47-66, 1938.

[30] J. von Neumann, Zur allgemeinen Theorie des Masses, Fund. Math., vol. 13, pp. 73-111, 1929.

[31] S. Wolfenstetter, Weakly Almost Periodic Functions on Hypergroups, Monatshefte Math. 96, pp. 67-79, 1983.