On the solvability of some systems of integro-differential equations with and without a drift

Messoud Efendiev[1,2], Vitali Vougalter[3 *]

[1] Helmholtz Zentrum München, Institut für Computational Biology, Ingolstädter Landstrasse 1
Neuherberg, 85764, Germany
e-mail: messoud.efendiyev@helmholtz-muenchen.de
[2] Department of Mathematics, Marmara University, Istanbul, Türkiye
e-mail: m.efendiyev@marmara.edu.tr

[3 *] Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract. We prove the existence of solutions for some integro-differential systems containing equations with and without the drift terms in the H^2 spaces by virtue of the fixed point technique when the elliptic equations contain second order differential operators with and without the Fredholm property, on the whole real line or on a finite interval with periodic boundary conditions. Let us emphasize that the study of the system case is more complicated than of the scalar situation and requires to overcome more cumbersome technicalities.

Keywords: solvability conditions, non-Fredholm operators, integro-differential systems, drift terms
AMS subject classification: 35J61, 35R09, 35K57

1 Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of the Fredholm operators are broadly used in many methods of the linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [2], [9], [19], [21]). This is the main result of the theory of the linear

elliptic problems. In the case of the unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^\alpha(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the assumptions stated above, the limiting operators are invertible (see [22]). In some simple cases, the limiting operators can be constructed explicitly. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at the infinities,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are given by:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients here are the constants, the essential spectrum of the operator, that is the set of the complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be found explicitly by means of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

The invertibility of the limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.
In the cases of the general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.
In the situations with non-Fredholm operators, the usual solvability conditions may not be applicable and the solvability relations are, in general, not known. There are some classes of operators for which the solvability conditions are obtained. We illustrate that with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. Clearly, the operator L coincides with its limiting operators. The homogeneous problem admits a nonzero bounded solution. Thus the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability relations can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a unique solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [28]). Here $S^d_{\sqrt{a}}$ denotes the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Thus, though the operator fails to satisfy the Fredholm property, the solvability conditions are formulated similarly. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

the Fourier transform is not directly applicable. Nevertheless, the solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [25]). As before, the solvability relations are formulated in terms of the orthogonality to the solutions of the homogeneous adjoint problem. There are several other examples of the linear elliptic non-Fredholm operators for which the solvability conditions can be obtained (see [13], [22], [23], [25], [27], [28]).

Solvability conditions play a crucial role in the analysis of the nonlinear elliptic equations. In the case of the operators without Fredholm property, in spite of some progress in the understanding of the linear problems, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [7], [8], [12], [13], [28]).

In the present article we study another class of stationary nonlinear systems of equations, for which the Fredholm property may or may not be satisfied:

$$\frac{d^2u_k}{dx^2} + b_k\frac{du_k}{dx} + a_k u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \quad 1 \le k \le K, \tag{1.2}$$

$$\frac{d^2u_k}{dx^2} + a_k u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \quad K+1 \le k \le N, \tag{1.3}$$

with the constants $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ and $K \ge 2$. Here $x \in \Omega \subseteq \mathbb{R}$. In the first case we consider the situation when $\Omega = \mathbb{R}$, such that $N \ge 4$. In the second part of the article we discuss the case of the finite interval $\Omega = I := [0, 2\pi]$ with periodic boundary conditions, so that $N \ge 5$. Throughout the work the vector function

$$u := (u_1, u_2, ..., u_N)^T \in \mathbb{R}^N. \tag{1.4}$$

For the simplicity of the presentation we restrict ourselves to the one dimensional case (the multidimensional case will be considered in our forthcoming paper). Article [12] is devoted to the studies of a single integro-differential equation with a drift term and the case without a transport term was covered in [26]. In the population dynamics the integro-differential equations describe the models with the intra-specific competition and the nonlocal consumption of resources (see e.g. [3], [4], [16]). The studies of the systems of integro-differential equations are of interest to us in the context of the complicated biological systems, where $u_k(x,t),\ k = 1, ..., N$ stand for the cell densities for various groups of cells in the organism. Let us use the explicit form of the solvability relations and study the existence of solutions of such nonlinear systems. We would like to emphasize especially that the solutions of the integro-differential equations with the drift terms are relevant to the understanding of the emergence and propagation of patterns in the theory of speciation (see [24]). The solvability of the linear problems involving the Laplace operator with the drift term was treated in [27], see also [5]. Standing lattice solitons in the discrete NLS equation with saturation were discussed in [1]. Fredholm structures, topological invariants and applications were covered in [9]. The work [10] deals with the finite and infinite dimensional attractors for the evolution equations of mathematical physics. The large time behavior of solutions of fourth order parabolic equations and ε-entropy of their attractors were analyzed in [11]. The articles [14] and [20] are crucial for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The work [15] is devoted to the exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations.

2 Formulation of the results

Our technical conditions are analogous to the ones of the [12], adapted to the work with vector functions. It is also more difficult to work in the Sobolev space $H^2(\Omega, \mathbb{R}^N)$, especially when Ω is a finite interval with periodic boundary conditions with the constraints applied. The nonlinear parts of system (1.2), (1.3) will satisfy the following regularity conditions.

Assumption 2.1. *Let $1 \leq k \leq N$. Functions $F_k(u,x) : \mathbb{R}^N \times \Omega \to \mathbb{R}$ are satisfying the Caratheodory condition (see [18]), such that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \leq \mathcal{K}|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega \tag{2.1}$$

with a constant $\mathcal{K} > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Furthermore, they are Lipschitz continuous functions, such that for any $u^{(1),(2)} \in \mathbb{R}^N$, $x \in \Omega$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)},x) - F_k(u^{(2)},x))^2} \leq L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N}, \tag{2.2}$$

with a constant $L > 0$.
In the case of $\Omega = I$ we assume that $F_k(u,0) = F_k(u,2\pi)$ for $u \in \mathbb{R}^N$ and all $1 \leq k \leq N$.

Here and further down the norm of a vector function given by (1.4) is

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

Note that the solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was considered in [6], where the nonlinear function was allowed to have a sublinear growth.

In order to study the existence of solutions of (1.2), (1.3), we introduce the auxiliary system of equations as

$$-\frac{d^2u_k}{dx^2}-b_k\frac{du_k}{dx}-a_ku_k=\int_\Omega G_k(x-y)F_k(v_1(y),v_2(y),...,v_N(y),y)dy, \quad 1\le k\le K, \tag{2.3}$$

$$-\frac{d^2u_k}{dx^2}-a_ku_k=\int_\Omega G_k(x-y)F_k(v_1(y),v_2(y),...,v_N(y),y)dy, \quad K+1\le k\le N, \tag{2.4}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ are the constants and $K \ge 2$.
Let us designate

$$(f_1(x),f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx, \tag{2.5}$$

with a slight abuse of notations when these functions are not square integrable, like for instance those involved in orthogonality relation (4.8) below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x)$ is bounded, the integral in the right side of (2.5) makes sense.
Let us first consider the situation of the whole real line, such that $\Omega = \mathbb{R}$. The appropriate Sobolev space is equipped with the norm

$$\|\phi\|^2_{H^2(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2\phi}{dx^2}\right\|^2_{L^2(\mathbb{R})}. \tag{2.6}$$

For a vector function with real valued components given by (1.4), we have

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^N \|u_k\|^2_{H^2(\mathbb{R})} = \sum_{k=1}^N \left\{\|u_k\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^2u_k}{dx^2}\right\|^2_{L^2(\mathbb{R})}\right\}. \tag{2.7}$$

We also use the norm

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^N \|u_k\|^2_{L^2(\mathbb{R})}.$$

By means of Assumption 2.1 above, we are not allowed to consider the higher powers of the nonlinearities, than the first one, which is restrictive from the point of view of the applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\Omega,\mathbb{R}^N)$ to $L^2(\Omega,\mathbb{R}^N)$.
The main issue for the system above is that in the absence of the drift terms we are dealing with the self-adjoint, non-Fredholm operators

$$-\frac{d^2}{dx^2}-a_k : H^2(\mathbb{R}) \to L^2(\mathbb{R}),\ a_k \ge 0,$$

which is the obstacle to solve our problem. The similar situations but in linear problems, both self- adjoint and non-self-adjoint involving the differential operators without the Fredholm property have been studied extensively in recent years (see [13], [22], [23], [25], [27], [28]).

However, the situation differs when the constants in the drift terms $b_k \neq 0$. For $1 \leq k \leq K$, the operators

$$L_{a,\ b,\ k} := -\frac{d^2}{dx^2} - b_k \frac{d}{dx} - a_k : \quad H^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.8}$$

with $a_k \geq 0$ and $b_k \in \mathbb{R},\ b_k \neq 0$ involved in the left side of (2.3) are non-self-adjoint.
By means of the standard Fourier transform, it can be easily verified that the essential spectra of the operators $L_{a,\ b,\ k}$ are given by

$$\lambda_{a,\ b,\ k}(p) = p^2 - a_k - ib_k p, \quad p \in \mathbb{R}.$$

Obviously, when $a_k > 0$ the operators $L_{a,\ b,\ k}$ are Fredholm, because their essential spectra stay away from the origin. But when $a_k = 0$ our operators $L_{a,\ b,\ k}$ fail to satisfy the Fredholm property since the origin belongs to their essential spectra.
We manage to establish that under the reasonable technical assumptions system (2.3), (2.4) defines a map $T_{a,b} : H^2(\mathbb{R}, \mathbb{R}^N) \to H^2(\mathbb{R}, \mathbb{R}^N)$, which is a strict contraction.

Theorem 2.2. *Let $\Omega = \mathbb{R},\ N \geq 4,\ K \geq 2,\ 1 \leq l \leq K-1,\ K+1 \leq r \leq N-1$, the integral kernels $G_k(x) : \mathbb{R} \to \mathbb{R},\ G_k(x) \in L^1(\mathbb{R})$ for all $1 \leq k \leq N$ and Assumption 2.1 holds.*

a) Let $a_k > 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ for $1 \leq k \leq l$.

b) Let $a_k = 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ for $l+1 \leq k \leq K$, additionally $xG_k(x) \in L^1(\mathbb{R})$ and orthogonality conditions (4.8) hold.

c) Let $a_k > 0,\ xG_k(x) \in L^1(\mathbb{R})$ for $K+1 \leq k \leq r$ and orthogonality relations (4.9) hold.

d) Let $a_k = 0,\ x^2G_k(x) \in L^1(\mathbb{R})$ for $r+1 \leq k \leq N$, orthogonality conditions (4.10) hold and $2\sqrt{\pi}QL < 1$ with Q defined in (4.7) below. Then the map $v \mapsto T_{a,b}v = u$ on $H^2(\mathbb{R}, \mathbb{R}^N)$ defined by problem (2.3), (2.4) has a unique fixed point $v^{(a,b)}$, which is the only solution of the system of equations (1.2), (1.3) in $H^2(\mathbb{R}, \mathbb{R}^N)$.

The fixed point $v^{(a,b)}$ is nontrivial provided that for some $1 \leq k \leq N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.

Let us note that in the case a) of the theorem above, when $a_k > 0,\ b_k \in \mathbb{R},\ b_k \neq 0$, the orthogonality relations are not needed.
In the second part of the article we study the analogous system on the finite interval $I = [0, 2\pi]$ with periodic boundary conditions with $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0,\ K \geq 2,\ N \geq 5$, namely for $1 \leq k \leq K$

$$\frac{d^2u_k}{dx^2} + b_k\frac{du_k}{dx} + a_k u_k + \int_0^{2\pi} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \tag{2.9}$$

and for $K+1 \leq k \leq N$

$$\frac{d^2u_k}{dx^2} + a_k u_k + \int_0^{2\pi} G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0. \tag{2.10}$$

Let us use the function space

$$H^2(I) := \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I),\ v(0) = v(2\pi),\ v'(0) = v'(2\pi)\} \tag{2.11}$$

aiming at $u_k(x) \in H^2(I)$ for $1 \le k \le l$ and $K+1 \le k \le r$ with $1 \le l \le K-1$ and $K+1 \le r \le q-1$ (see Theorem 2.3 and Lemma A2 below).
For the technical purposes we introduce the auxiliary constrained subspaces

$$H_k^2(I) := \Big\{v \in H^2(I) \ \Big| \ \Big(v(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\Big)_{L^2(I)} = 0\Big\}, \quad n_k \in \mathbb{N}, \quad r+1 \le k \le q. \tag{2.12}$$

Our goal is to have $u_k(x) \in H_k^2(I),\ r+1 \le k \le q$, where $r+1 \le q \le N-1$. Also,

$$H_0^2(I) := \{v \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0\}. \tag{2.13}$$

We plan to have $u_k(x) \in H_0^2(I)$ for $l+1 \le k \le K$ and $q+1 \le k \le N$ (see Theorem 2.3 and Lemma A2). The constrained subspaces (2.12) and (2.13) are Hilbert spaces as well (see e.g. Chapter 2.1 of [17]).
The resulting space used for demonstrating the existence of the solution $u(x) : I \to \mathbb{R}^N$ of the system of equations (2.9), (2.10) will be the direct sum of the spaces introduced above, namely

$$H_c^2(I, \mathbb{R}^N) := \oplus_{k=1}^{l} H^2(I) \oplus_{k=l+1}^{K} H_0^2(I) \oplus_{k=K+1}^{r} H^2(I) \oplus_{k=r+1}^{q} H_k^2(I) \oplus_{k=q+1}^{N} H_0^2(I). \tag{2.14}$$

The corresponding Sobolev norm will be

$$\|u\|^2_{H_c^2(I,\mathbb{R}^N)} := \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(I)} + \|u_k''\|^2_{L^2(I)}\}, \tag{2.15}$$

where $u(x) : I \to \mathbb{R}^N$.
Let us demonstrate that under the reasonable technical conditions system (2.3), (2.4) with $\Omega = I$ defines a map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$, which is a strict contraction.

Theorem 2.3. *Let $\Omega = I,\ N \ge 5,\ K \ge 2,\ 1 \le l \le K-1,\ K+1 \le r \le q-1,\ r+1 \le q \le N-1$, the integral kernels $G_k(x) : I \to \mathbb{R},\ G_k(x) \in C(I),\ G_k(0) = G_k(2\pi)$ for all $1 \le k \le N$ and Assumption 2.1 is valid.*

a) Let $a_k > 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ for $1 \le k \le l$.

b) Let $a_k = 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ for $l+1 \le k \le K$ and orthogonality condition (4.28) holds.

c) Let $a_k > 0,\ a_k \ne n^2,\ n \in \mathbb{Z}$ for $K+1 \le k \le r$.

d) Let $a_k = n_k^2,\ n_k \in \mathbb{N}$ for $r+1 \le k \le q$ and orthogonality relations (4.29) are valid.

e) Let $a_k = 0$ for $q+1 \le k \le N$, orthogonality condition (4.28) holds and $2\sqrt{\pi}\mathcal{Q}L < 1$, where $\mathcal{Q}$ is introduced in (4.27). Then the map $v \mapsto \tau_{a,b} v = u$ on $H_c^2(I, \mathbb{R}^N)$ defined by the *system of equations (2.3), (2.4) has a unique fixed point $v^{(a,b)}$, which is the only solution of system (2.9), (2.10) in $H_c^2(I, \mathbb{R}^N)$.*

The fixed point $v^{(a,b)}$ does not vanish identically in I provided that for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$ the Fourier coefficients $G_{k,n}F_k(0,x)_n \neq 0$.

Remark 2.4. *Note that in the present work we deal with real valued vector functions by means of the assumptions on $F_k(u,x)$ and $G_k(x)$ involved in the nonlocal terms of the systems of equations considered above.*

3 The existence of solutions for the integro-differential systems

Proof of Theorem 2.2. First we suppose that for a certain $v \in H^2(\mathbb{R}, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)} \in H^2(\mathbb{R}, \mathbb{R}^N)$ of problem (2.3), (2.4). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ will be a solution of the homogeneous system of equations

$$-\frac{d^2 w_k}{dx^2} - b_k \frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \le k \le K,$$

$$-\frac{d^2 w_k}{dx^2} - a_k w_k = 0, \quad K+1 \le k \le N.$$

But the operators $-\frac{d^2}{dx^2} - a_k$, $L_{a,\ b,\ k} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $L_{a,\ b,\ k}$ is defined in (2.8) do not have any nontrivial zero modes. Therefore, $w(x)$ vanishes on $\mathbb{R}$.
Let us choose an arbitrary $v(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$. We apply the standard Fourier transform (4.1) to both sides of system (2.3), (2.4). This gives us

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k - ib_k p}, \quad p^2\widehat{u_k}(p) = \sqrt{2\pi}\frac{p^2\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k - ib_k p}, \quad 1 \le k \le K, \tag{3.1}$$

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k}, \quad p^2\widehat{u_k}(p) = \sqrt{2\pi}\frac{p^2\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k}, \quad K+1 \le k \le N. \tag{3.2}$$

Here $\widehat{f_k}(p)$ stands for the Fourier image of $F_k(v(x), x)$.
Clearly, for $1 \le k \le K$, we have the upper bounds with $N_{a,\ b,\ k}$ defined in (4.3), namely

$$|\widehat{u_k}(p)| \le \sqrt{2\pi}N_{a,\ b,\ k}|\widehat{f_k}(p)| \quad and \quad |p^2\widehat{u_k}(p)| \le \sqrt{2\pi}N_{a,\ b,\ k}|\widehat{f_k}(p)|.$$

For $K+1 \le k \le N$, we obtain

$$|\widehat{u_k}(p)| \le \sqrt{2\pi}M_{a,\ k}|\widehat{f_k}(p)| \quad and \quad |p^2\widehat{u_k}(p)| \le \sqrt{2\pi}M_{a,\ k}|\widehat{f_k}(p)|,$$

with $M_{a,\ k}$ introduced in (4.4). Note that $N_{a,\ b,\ k} < \infty$ by virtue of Lemma A1 of the Appendix without any orthogonality conditions for $a_k > 0,\ 1 \le k \le l$ and under orthogonality relation (4.8) when $a_k = 0,\ l+1 \le k \le K$. Also, $M_{a,\ k} < \infty$ under orthogonality conditions (4.9) when $a_k > 0,\ K+1 \le k \le r$ and under orthogonality relations (4.10) for $a_k = 0,\ r+1 \le k \le N$.
This allows us to obtain the upper bound on the norm as

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} = \sum_{k=1}^{N} \{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})}\} \le$$

$$\le 4\pi \sum_{k=1}^{K} N^2_{a,\ b,\ k} \|F_k(v(x),x)\|^2_{L^2(\mathbb{R})} + 4\pi \sum_{k=K+1}^{N} M^2_{a,\ k} \|F_k(v(x),x)\|^2_{L^2(\mathbb{R})}. \tag{3.3}$$

The right side of (3.3) is finite via (2.1) of Assumption 2.1 since $|v(x)|_{\mathbb{R}^N} \in L^2(\mathbb{R})$. Thus, for an arbitrary $v(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ there exists a unique solution $u(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ of problem (2.3), (2.4). Its Fourier image is given by (3.1), (3.2). Therefore, the map $T_{a,b} : H^2(\mathbb{R},\mathbb{R}^N) \to H^2(\mathbb{R},\mathbb{R}^N)$ is well defined.
This enables us to choose arbitrarily $v^{(1),(2)}(x) \in H^2(\mathbb{R},\mathbb{R}^N)$, such that under the given conditions their images $u^{(1),(2)} := T_{a,b}v^{(1),(2)} \in H^2(\mathbb{R},\mathbb{R}^N)$. By means of (2.3), (2.4) along with (4.1),

$$\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{p^2 - a_k - ib_kp}, \quad \widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{p^2 - a_k - ib_kp}, \quad 1 \le k \le K,$$

$$\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{p^2 - a_k}, \quad \widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{p^2 - a_k}, \quad K+1 \le k \le N.$$

Here $\widehat{f_k^{(1)}}(p)$ and $\widehat{f_k^{(2)}}(p)$ denote the Fourier transforms of $F_k(v^{(1)}(x),x)$ and $F_k(v^{(2)}(x),x)$ respectively.
Hence, for $1 \le k \le K$, we easily derive

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi}N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi}N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|$$

and for $K+1 \le k \le N$

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi}M_{a,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi}M_{a,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|.$$

Then for the appropriate norm of the difference of vector functions we obtain

$$\|u^{(1)}-u^{(2)}\|^2_{H^2(\mathbb{R},\mathbb{R}^N)}=\sum_{k=1}^N\left\{\left\|\widehat{u_k^{(1)}}(p)-\widehat{u_k^{(2)}}(p)\right\|^2_{L^2(\mathbb{R})}+\left\|p^2\widehat{u_k^{(1)}}(p)-p^2\widehat{u_k^{(2)}}(p)\right\|^2_{L^2(\mathbb{R})}\right\}\leq$$

$$\leq 4\pi Q^2\sum_{k=1}^N\|F_k(v^{(1)}(x),x)-F_k(v^{(2)}(x),x)\|^2_{L^2(\mathbb{R})},$$

where Q is defined in (4.7). Clearly, all $v_k^{(1),(2)}(x)\in H^2(\mathbb{R})\subset L^\infty(\mathbb{R})$ due to the Sobolev embedding.
Condition (2.2) gives us

$$\sum_{k=1}^N\|F_k(v^{(1)}(x),x)-F_k(v^{(2)}(x),x)\|^2_{L^2(\mathbb{R})}\leq L^2\|v^{(1)}-v^{(2)}\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}.$$

Thus,

$$\|T_{a,b}v^{(1)}-T_{a,b}v^{(2)}\|_{H^2(\mathbb{R},\mathbb{R}^N)}\leq 2\sqrt{\pi}QL\|v^{(1)}-v^{(2)}\|_{H^2(\mathbb{R},\mathbb{R}^N)}. \tag{3.4}$$

The constant in the right side of (3.4) is less than one via the one of our assumptions. Therefore, by virtue of the Fixed Point Theorem, there exists a unique vector function $v^{(a,b)}\in H^2(\mathbb{R},\mathbb{R}^N)$, such that $T_{a,b}v^{(a,b)}=v^{(a,b)}$, which is the only solution of problem (1.2), (1.3) in $H^2(\mathbb{R},\mathbb{R}^N)$. Suppose $v^{(a,b)}(x)$ vanishes identically on the real line. This will contradict to our assumption that for a certain $1\leq k\leq N$, the Fourier transforms of $G_k(x)$ and $F_k(0,x)$ are nontrivial on a set of nonzero Lebesgue measure in $\mathbb{R}$. ■

Then we turn our attention to establishing the existence of solutions for our system of integro-differential equations on the finite interval with periodic boundary conditions.

Proof of Theorem 2.3. Let us first suppose that for some $v\in H_c^2(I,\mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}\in H_c^2(I,\mathbb{R}^N)$ of system (2.3), (2.4) with $\Omega=I$. Then the difference $\tilde{w}(x):=u^{(1)}(x)-u^{(2)}(x)\in H_c^2(I,\mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$-\frac{d^2\tilde{w}_k}{dx^2}-b_k\frac{d\tilde{w}_k}{dx}-a_k\tilde{w}_k=0,\quad 1\leq k\leq K, \tag{3.5}$$

$$-\frac{d^2\tilde{w}_k}{dx^2}-a_k\tilde{w}_k=0,\quad K+1\leq k\leq N. \tag{3.6}$$

Evidently, each operator contained in the left side of system (3.5)

$$\mathcal{L}_{a,\ b,\ k}:=-\frac{d^2}{dx^2}-b_k\frac{d}{dx}-a_k:\quad H^2(I)\to L^2(I), \tag{3.7}$$

where $1\leq k\leq l,\ a_k>0,\ b_k\in\mathbb{R},\ b_k\neq 0$ is Fredholm, non-self-adjoint.

Its set of eigenvalues is

$$\lambda_{a,\ b,\ k}(n) = n^2 - a_k - ib_k n, \quad n \in \mathbb{Z}. \tag{3.8}$$

The corresponding eigenfunctions are the standard Fourier harmonics

$$\frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}. \tag{3.9}$$

When $l+1 \le k \le K$, we have $a_k = 0,\ b_k \in \mathbb{R},\ b_k \neq 0$. Let us deal with the operators

$$\mathcal{L}_{0,\ b,\ k} := -\frac{d^2}{dx^2} - b_k \frac{d}{dx} : \quad H_0^2(I) \to L^2(I) \tag{3.10}$$

involved in system (3.5).
Obviously, each operator (3.10) has the eigenvalues given by formula (3.8) with $a_k = 0,\ b_k \in \mathbb{R},\ b_k \neq 0,\ n \in \mathbb{Z},\ n \neq 0$.
The corresponding eigenfunctions in this case are (3.9) with $n \in \mathbb{Z},\ n \neq 0$.
Clearly, every operator contained in the left side of the system of equations (3.6)

$$\mathcal{L}_{a,\ 0,\ k} := -\frac{d^2}{dx^2} - a_k : \quad H^2(I) \to L^2(I) \tag{3.11}$$

with $K+1 \le k \le r,\ a_k > 0,\ a_k \neq n^2,\ n \in \mathbb{Z}$ is Fredholm, self-adjoint.
Its set of eigenvalues is

$$\lambda_{a,\ 0,\ k}(n) = n^2 - a_k, \quad n \in \mathbb{Z}. \tag{3.12}$$

The corresponding eigenfunctions are given by (3.9).
For $r+1 \le k \le q$, we have $a_k = n_k^2,\ n_k \in \mathbb{N}$. Let us consider the operators

$$\mathcal{L}_{n_k^2,\ 0,\ k} := -\frac{d^2}{dx^2} - n_k^2 : \quad H_k^2(I) \to L^2(I) \tag{3.13}$$

involved in the left side of system (3.6).
The eigenvalues of each operator (3.13) are written in (3.12) with $a_k = n_k^2,\ n \in \mathbb{Z},\ n \neq \pm n_k$.
The corresponding eigenfunctions are given by formula (3.9) with $n \in \mathbb{Z},\ n \neq \pm n_k$.
When $q+1 \le k \le N$, all the constants a_k are trivial. Then the operator contained in the left side of the system of equations (3.6) in this situation is

$$\mathcal{L}_{0,\ 0,\ k} := -\frac{d^2}{dx^2} : \quad H_0^2(I) \to L^2(I). \tag{3.14}$$

Its eigenvalues are

$$\lambda_{0,\ 0,\ k}(n) = n^2, \quad n \in \mathbb{Z}, \quad n \neq 0. \tag{3.15}$$

We have the corresponding eigenfunctions written in (3.9) with $n \in \mathbb{Z},\ n \neq 0$.

Note that all the operators mentioned above, which are involved in the left side of the homogeneous system (3.5), (3.6) have the trivial kernels. Thus, the vector function $\tilde{w}(x)$ vanishes identically on the interval I.
Let us choose arbitrarily $v(x) \in H_c^2(I, \mathbb{R}^N)$ and apply the Fourier transform (4.21) to both sides of the system of equations (2.3), (2.4) with $\Omega = I$. This yields

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n}f_{k,n}}{n^2 - a_k - ib_k n}, \quad n^2 u_{k,n} = \sqrt{2\pi}\frac{n^2 G_{k,n}f_{k,n}}{n^2 - a_k - ib_k n}, \quad 1 \le k \le K, \quad n \in \mathbb{Z}, \tag{3.16}$$

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n}f_{k,n}}{n^2 - a_k}, \quad n^2 u_{k,n} = \sqrt{2\pi}\frac{n^2 G_{k,n}f_{k,n}}{n^2 - a_k}, \quad K+1 \le k \le N, \quad n \in \mathbb{Z}, \tag{3.17}$$

where $f_{k,n} := F_k(v(x), x)_n$.
From (3.16) and (3.17) we easily obtain that

$$|u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}|, \quad |n^2 u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}|, \quad 1 \le k \le K, \quad n \in \mathbb{Z}, \tag{3.18}$$

$$|u_{k,n}| \le \sqrt{2\pi}\mathcal{M}_{a,\ k}|f_{k,n}|, \quad |n^2 u_{k,n}| \le \sqrt{2\pi}\mathcal{M}_{a,\ k}|f_{k,n}|, \quad K+1 \le k \le N, \quad n \in \mathbb{Z}. \tag{3.19}$$

In the estimates above we have $\mathcal{N}_{a,\ b,\ k}$ introduced in (4.23) and $\mathcal{M}_{a,\ k}$ was defined in (4.24). Clearly, $\mathcal{N}_{a,\ b,\ k} < \infty$ by means of Lemma A2 of the Appendix without any orthogonality relations for $a_k > 0,\ 1 \le k \le l$ and under orthogonality condition (4.28) when $a_k = 0,\ l+1 \le k \le K$. Similarly, $\mathcal{M}_{a,\ k} < \infty$ for $a_k > 0,\ a_k \ne n^2,\ n \in \mathbb{Z},\ K+1 \le k \le r$, under orthogonality relations (4.29) when $a_k = n_k^2,\ n_k \in \mathbb{N},\ r+1 \le k \le q$ and under orthogonality condition (4.28) for $a_k = 0,\ q+1 \le k \le N$ by virtue of Lemma A2.
By means of (3.18) and (3.19), we derive

$$\|u\|^2_{H_c^2(I,\mathbb{R}^N)} = \sum_{k=1}^{K}\Big[\sum_{n=-\infty}^{\infty}|u_{k,n}|^2 + \sum_{n=-\infty}^{\infty}|n^2 u_{k,n}|^2\Big] + \sum_{k=K+1}^{N}\Big[\sum_{n=-\infty}^{\infty}|u_{k,n}|^2 + \sum_{n=-\infty}^{\infty}|n^2 u_{k,n}|^2\Big] \le$$

$$\le 4\pi\sum_{k=1}^{K}\mathcal{N}^2_{a,\ b,\ k}\|F_k(v(x),x)\|^2_{L^2(I)} + 4\pi\sum_{k=K+1}^{N}\mathcal{M}^2_{a,\ k}\|F_k(v(x),x)\|^2_{L^2(I)}. \tag{3.20}$$

Let us recall inequality (2.1) of Assumption 2.1. We have here $|v(x)|_{\mathbb{R}^N} \in L^2(I)$, such that all $F_k(v(x), x) \in L^2(I)$ and the right side of (3.20) is finite. Hence, for any $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exists a unique $u(x) \in H_c^2(I, \mathbb{R}^N)$, which solves system (2.3), (2.4) with $\Omega = I$. Its Fourier transform is given by formulas (3.16) and (3.17). Thus, the map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$ is well defined.
Let us choose arbitrarily $v^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$. Under the stated assumptions, their images under the map discussed above are $u^{(1),(2)} := \tau_{a,b}v^{(1),(2)} \in H_c^2(I, \mathbb{R}^N)$. By virtue of (2.3) and (2.4) with $\Omega = I$ along with (4.21), we arrive at

$$u^{(1)}_{k,n} = \sqrt{2\pi}\frac{G_{k,n}f^{(1)}_{k,n}}{n^2 - a_k - ib_k n}, \quad u^{(2)}_{k,n} = \sqrt{2\pi}\frac{G_{k,n}f^{(2)}_{k,n}}{n^2 - a_k - ib_k n}, \quad 1 \le k \le K, \quad n \in \mathbb{Z},$$

$$u_{k,n}^{(1)} = \sqrt{2\pi}\frac{G_{k,n}f_{k,n}^{(1)}}{n^2 - a_k}, \quad u_{k,n}^{(2)} = \sqrt{2\pi}\frac{G_{k,n}f_{k,n}^{(2)}}{n^2 - a_k}, \quad K+1 \le k \le N, \quad n \in \mathbb{Z}.$$

Here $f_{k,n}^{(1)}$ and $f_{k,n}^{(2)}$ stand for the Fourier images of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively under transform (4.21).
Thus, for $1 \le k \le K,\ n \in \mathbb{Z}$, we have

$$|u_{k,n}^{(1)} - u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|, \quad |n^2 u_{k,n}^{(1)} - n^2 u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{N}_{a,\ b,\ k}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|.$$

Similarly, for $K+1 \le k \le N,\ n \in \mathbb{Z}$, we obtain

$$|u_{k,n}^{(1)} - u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{M}_{a,\ k}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|, \quad |n^2 u_{k,n}^{(1)} - n^2 u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{M}_{a,\ k}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|.$$

Let us estimate the appropriate norm of the difference of the vector functions as

$$\|u^{(1)} - u^{(2)}\|_{H_c^2(I,\mathbb{R}^N)}^2 = \sum_{k=1}^{K}\Big[\sum_{n=-\infty}^{\infty}|u_{k,n}^{(1)} - u_{k,n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty}|n^2 u_{k,n}^{(1)} - n^2 u_{k,n}^{(2)}|^2\Big] +$$

$$+ \sum_{k=K+1}^{N}\Big[\sum_{n=-\infty}^{\infty}|u_{k,n}^{(1)} - u_{k,n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty}|n^2 u_{k,n}^{(1)} - n^2 u_{k,n}^{(2)}|^2\Big] \le$$

$$\le 4\pi\mathcal{Q}^2\sum_{k=1}^{N}\|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(I)}^2,$$

with $\mathcal{Q}$ introduced in (4.27). Evidently, all $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I)$ via the Sobolev embedding.
We recall condition (2.2) of Assumption 2.1, such that

$$\sum_{k=1}^{N}\|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(I)}^2 \le L^2\|v^{(1)} - v^{(2)}\|_{H_c^2(I,\mathbb{R}^N)}^2.$$

Hence,

$$\|\tau_{a,b}v^{(1)} - \tau_{a,b}v^{(2)}\|_{H_c^2(I,\mathbb{R}^N)} \le 2\sqrt{\pi}\mathcal{Q}L\|v^{(1)} - v^{(2)}\|_{H_c^2(I,\mathbb{R}^N)}. \tag{3.21}$$

The constant in the right side of bound (3.21) is less than one as assumed. By means of the Fixed Point Theorem, there exists a unique vector function $v^{(a,b)} \in H_c^2(I,\mathbb{R}^N)$, so that $\tau_{a,b}v^{(a,b)} = v^{(a,b)}$. This is the only solution of the system of equations (2.9), (2.10) in $H_c^2(I,\mathbb{R}^N)$. Let us suppose that $v^{(a,b)}(x)$ vanishes identically in I. This will contradict to the given condition that for a certain $1 \le k \le N$ and some $n \in \mathbb{Z}$ the Fourier coefficients $G_{k,n}F_k(0,x)_n \ne 0$. ■

4 Appendix

Let $G_k(x)$ be a function, $G_k(x) : \mathbb{R} \to \mathbb{R}$, for which we designate its standard Fourier transform using the hat symbol as

$$\widehat{G_k}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_k(x) e^{-ipx} dx, \quad p \in \mathbb{R}. \tag{4.1}$$

Clearly

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|G_k\|_{L^1(\mathbb{R})} \tag{4.2}$$

and $G_k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G_k}(q) e^{iqx} dq,\ x \in \mathbb{R}$. For the technical purposes we introduce the auxiliary quantities

$$N_{a,\ b,\ k} := \max\left\{ \left\| \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})}, \quad \left\| \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} \right\}, \quad 1 \le k \le K, \tag{4.3}$$

$$M_{a,\ k} := \max\left\{ \left\| \frac{\widehat{G_k}(p)}{p^2 - a_k} \right\|_{L^\infty(\mathbb{R})}, \quad \left\| \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k} \right\|_{L^\infty(\mathbb{R})} \right\}, \quad K+1 \le k \le N, \tag{4.4}$$

where $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ are the constants and $K \ge 2,\ N \ge 4$. Let $N_{0,\ b,\ k}$ denote (4.3) when a_k vanishes and $M_{0,\ k}$ stands for (4.4) when $a_k = 0$. Under the assumptions of Lemma A1 below, quantities (4.3) and (4.4) will be finite. This will allow us to define

$$N_{a,\ b} := \max_{1 \le k \le K} N_{a,\ b,\ k} < \infty, \tag{4.5}$$

$$M_a := \max_{K+1 \le k \le N} M_{a,\ k} < \infty \tag{4.6}$$

and

$$Q := \max\{N_{a,\ b},\ M_a\}. \tag{4.7}$$

The auxiliary lemma below is the adaptation of the ones established in [12] and [26] for the studies of the single integro-differential equation with and without a drift. Let us present it for the convenience of the readers.

Lemma A1. *Let $N \ge 4,\ K \ge 2,\ 1 \le l \le K-1,\ K+1 \le r \le N-1$, the integral kernels $G_k(x) : \mathbb{R} \to \mathbb{R},\ G_k(x) \in L^1(\mathbb{R})$ for all $1 \le k \le N$.*

a) Let $a_k > 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ for $1 \le k \le l$. Then $N_{a,\ b,\ k} < \infty$.

b) Let $a_k = 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ for $l+1 \le k \le K$ and additionally $xG_k(x) \in L^1(\mathbb{R})$. Then $N_{0,\ b,\ k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0 \tag{4.8}$$

holds.

c) Let $a_k > 0$ *and* $xG_k(x) \in L^1(\mathbb{R})$ *for* $K+1 \le k \le r$. *Then* $M_{a,\ k} < \infty$ *if and only if*

$$\left(G_k(x), \frac{e^{\pm i\sqrt{a_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \tag{4.9}$$

is valid.

d) Let $a_k = 0$ *and* $x^2 G_k(x) \in L^1(\mathbb{R})$ *for* $r+1 \le k \le N$. *Then* $M_{0,\ k} < \infty$ *if and only if*

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0 \quad \text{and} \quad (G_k(x), x)_{L^2(\mathbb{R})} = 0 \tag{4.10}$$

holds.

Proof. Note that in both cases a) and b) of our lemma the boundedness of $\frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$ implies the boundedness of $\frac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$. Indeed, we can express $\frac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$ as the following sum

$$\widehat{G_k}(p) + a_k \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} + ib_k \frac{p\widehat{G_k}(p)}{p^2 - a_k - ib_k p}. \tag{4.11}$$

Obviously, the first term in (4.11) is bounded by virtue of (4.2) because $G_k(x) \in L^1(\mathbb{R})$ due to the one of our assumptions. The third term in (4.11) can be trivially estimated from above in the absolute value by means of (4.2) as

$$\frac{|b_k||p||\widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}} \le \frac{1}{\sqrt{2\pi}} \|G_k(x)\|_{L^1(\mathbb{R})} < \infty.$$

Therefore, $\frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \in L^\infty(\mathbb{R})$ yields $\frac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \in L^\infty(\mathbb{R})$.
To establish the result of the part a) of our lemma, we need to estimate

$$\frac{|\widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}}. \tag{4.12}$$

Evidently, the numerator of (4.12) can be easily estimated from above by means of (4.2) and the denominator in (4.12) can be trivially bounded below by a finite, positive constant, such that

$$\left|\frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p}\right| \le C\|G_k(x)\|_{L^1(\mathbb{R})} < \infty.$$

Here and further down C will stand for a finite, positive constant. This implies that under our assumptions, when $a_k > 0$ we have $N_{a,\ b,\ k} < \infty$.
In the case b) of the lemma when $a_k = 0$, we use the identity

$$\widehat{G_k}(p) = \widehat{G_k}(0) + \int_0^p \frac{d\widehat{G_k}(s)}{ds} ds.$$

Hence
$$\frac{\widehat{G_k}(p)}{p^2-ib_kp}=\frac{\widehat{G_k}(0)}{p(p-ib_k)}+\frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds}ds}{p(p-ib_k)}. \tag{4.13}$$
By virtue of definition (4.1) of the standard Fourier transform, we easily obtain the upper bound
$$\left|\frac{d\widehat{G_k}(p)}{dp}\right|\le\frac{1}{\sqrt{2\pi}}\|xG_k(x)\|_{L^1(\mathbb{R})}. \tag{4.14}$$
We easily arrive at
$$\left|\frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds}ds}{p(p-ib_k)}\right|\le\frac{\|xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b_k|}<\infty$$
via our assumptions. Thus, the expression in the left side of (4.13) is bounded if and only if $\widehat{G_k}(0)$ vanishes, which is equivalent to orthogonality relation (4.8).
In the cases c) and d) of the lemma, we can write
$$\frac{p^2\widehat{G_k}(p)}{p^2-a_k}=\widehat{G_k}(p)+a_k\frac{\widehat{G_k}(p)}{p^2-a_k},$$
so that $\frac{\widehat{G_k}(p)}{p^2-a_k}\in L^\infty(\mathbb{R})$ implies that $\frac{p^2\widehat{G_k}(p)}{p^2-a_k}\in L^\infty(\mathbb{R})$ as well.
To demonstrate the validity of the result of the part c) of our lemma, we express
$$\frac{\widehat{G_k}(p)}{p^2-a_k}=\frac{\widehat{G_k}(p)}{p^2-a_k}\chi_{I_{\delta_k}^+}+\frac{\widehat{G_k}(p)}{p^2-a_k}\chi_{I_{\delta_k}^-}+\frac{\widehat{G_k}(p)}{p^2-a_k}\chi_{I_{\delta_k}^c}. \tag{4.15}$$
Here and below χ_A will denote the characteristic function of a set $A\subseteq\mathbb{R}$ and A^c will stand for its complement. The sets
$$I_{\delta_k}^+:=\{p\in\mathbb{R}\mid\sqrt{a_k}-\delta_k<p<\sqrt{a_K}+\delta_k\},\quad I_{\delta_k}^-:=\{p\in\mathbb{R}\mid-\sqrt{a_k}-\delta_k<p<-\sqrt{a_k}+\delta_k\}$$
with $0<\delta_k<\sqrt{a_k}$ and $I_{\delta_k}:=I_{\delta_k}^+\cup I_{\delta_k}^-$.
The third term in the right side of (4.15) can be trivially bounded from above in the absolute value by means of (4.2) by $\frac{1}{\sqrt{2\pi}\delta_k^2}\|G_k\|_{L^1(\mathbb{R})}<\infty$.
Clearly, we can write
$$\widehat{G_k}(p)=\widehat{G_k}(\sqrt{a_k})+\int_{\sqrt{a_k}}^p\frac{d\widehat{G_k}(s)}{ds}ds,\quad \widehat{G_k}(p)=\widehat{G_k}(-\sqrt{a_k})+\int_{-\sqrt{a_k}}^p\frac{d\widehat{G_k}(s)}{ds}ds.$$
Thus, the sum of the first two terms in the right side of (4.15) is given by
$$\frac{\widehat{G_k}(\sqrt{a_k})}{p^2-a_k}\chi_{I_{\delta_k}^+}+\frac{\widehat{G_k}(-\sqrt{a_k})}{p^2-a_k}\chi_{I_{\delta_k}^-}+\frac{\int_{\sqrt{a_k}}^p\frac{d\widehat{G_k}(s)}{ds}ds}{p^2-a_k}\chi_{I_{\delta_k}^+}+\frac{\int_{-\sqrt{a_k}}^p\frac{d\widehat{G_k}(s)}{ds}ds}{p^2-a_k}\chi_{I_{\delta_k}^-}.$$

By virtue of (4.14), we derive

$$\left|\frac{\int_{\sqrt{a_k}}^{p}\frac{d\widehat{G_k}(s)}{ds}ds}{p^2-a_k}\chi_{I_{\delta_k}^+}\right| \leq \frac{\|xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}(2\sqrt{a_k}-\delta_k)} < \infty,$$

$$\left|\frac{\int_{-\sqrt{a_k}}^{p}\frac{d\widehat{G_k}(s)}{ds}ds}{p^2-a_k}\chi_{I_{\delta_k}^-}\right| \leq \frac{\|xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}(2\sqrt{a_k}-\delta_k)} < \infty.$$

Therefore, it remains to consider

$$\frac{\widehat{G_k}(\sqrt{a_k})}{p^2-a_k}\chi_{I_{\delta_k}^+} + \frac{\widehat{G_k}(-\sqrt{a_k})}{p^2-a_k}\chi_{I_{\delta_k}^-}. \tag{4.16}$$

Evidently, (4.16) is bounded if and only if $\widehat{G_k}(\pm\sqrt{a_k}) = 0$. This is equivalent to the orthogonality conditions (4.9).
Finally, we turn our attention to the case d) of the lemma when $a_k = 0$, so that

$$\frac{\widehat{G_k}(p)}{p^2} = \frac{\widehat{G_k}(p)}{p^2}\chi_{\{|p|\leq 1\}} + \frac{\widehat{G_k}(p)}{p^2}\chi_{\{|p|>1\}}. \tag{4.17}$$

The second term in the right side of (4.17) can be easily estimated from above in the absolute value as

$$\left|\frac{\widehat{G_k}(p)}{p^2}\chi_{\{|p|>1\}}\right| \leq \|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R})} < \infty$$

via (4.2). Obviously,

$$\widehat{G_k}(p) = \widehat{G_k}(0) + p\frac{d\widehat{G_k}}{dp}(0) + \int_0^p\left(\int_0^s \frac{d^2\widehat{G_k}(q)}{dq^2}dq\right)ds,$$

such that the first term in right side of (4.17) equals to

$$\left[\frac{\widehat{G_k}(0)}{p^2} + \frac{\frac{d\widehat{G_k}}{dp}(0)}{p} + \frac{\int_0^p\left(\int_0^s \frac{d^2\widehat{G_k}(q)}{dq^2}dq\right)ds}{p^2}\right]\chi_{\{|p|\leq 1\}}. \tag{4.18}$$

Using definition (4.1) of the standard Fourier transform, we derive

$$\left|\frac{d^2\widehat{G_k}(p)}{dp^2}\right| \leq \frac{1}{\sqrt{2\pi}}\|x^2G_k(x)\|_{L^1(\mathbb{R})},$$

so that

$$\left|\frac{\int_0^p\left(\int_0^s \frac{d^2\widehat{G_k}(q)}{dq^2}dq\right)ds}{p^2}\chi_{\{|p|\leq 1\}}\right| \leq \frac{\|x^2G_k(x)\|_{L^1(\mathbb{R})}}{2\sqrt{2\pi}} < \infty$$

as assumed. Hence, it remains to analyze

$$\left[\frac{\widehat{G_k}(0)}{p^2} + \frac{\frac{d\widehat{G_k}}{dp}(0)}{p}\right]\chi_{\{|p|\le 1\}}. \tag{4.19}$$

From definition (4.1) of the standard Fourier transform we deduce that

$$\widehat{G_k}(0) = \frac{1}{\sqrt{2\pi}}(G_k(x), 1)_{L^2(\mathbb{R})}, \quad \frac{d\widehat{G_k}}{dp}(0) = -\frac{i}{\sqrt{2\pi}}(G_k(x), x)_{L^2(\mathbb{R})},$$

such that (4.19) is equal to

$$\frac{1}{\sqrt{2\pi}}\left[\frac{(G_k(x), 1)_{L^2(\mathbb{R})}}{p^2} - i\frac{(G_k(x), x)_{L^2(\mathbb{R})}}{p}\right]\chi_{\{|p|\le 1\}}. \tag{4.20}$$

Obviously, (4.20) belongs to $L^\infty(\mathbb{R})$ if and only if orthogonality relations (4.10) are valid. ■

Let the continuous function $G_k(x) : I \to \mathbb{R},\ G_k(0) = G_k(2\pi)$. Its Fourier transform on the finite interval is given by

$$G_{k,n} := \int_0^{2\pi} G_k(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}, \tag{4.21}$$

so that $G_k(x) = \sum_{n=-\infty}^{\infty} G_{k,n}\frac{e^{inx}}{\sqrt{2\pi}}$. Obviously, the inequalities

$$\|G_{k,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_k(x)\|_{L^1(I)}, \quad \|G_k(x)\|_{L^1(I)} \le 2\pi\|G_k(x)\|_{C(I)} \tag{4.22}$$

hold.
Similarly to the whole real line case, we define

$$\mathcal{N}_{a,\ b,\ k} := \max\left\{\left\|\frac{G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}\right\}, \quad 1 \le k \le K, \tag{4.23}$$

$$\mathcal{M}_{a,\ k} := \max\left\{\left\|\frac{G_{k,n}}{n^2 - a_k}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_{k,n}}{n^2 - a_k}\right\|_{l^\infty}\right\}, \quad K+1 \le k \le N \tag{4.24}$$

with the constants $a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$ and $K \ge 2,\ N \ge 5$. Let $\mathcal{N}_{0,\ b,\ k}$ stand for (4.23) when $a_k = 0$ and $\mathcal{M}_{0,\ k}$ denote (4.24) when a_k is trivial. Under the conditions of Lemma A2 below, expressions (4.23), (4.24) will be finite. This will enable us to introduce

$$\mathcal{N}_{a,\ b} := \max_{1\le k\le K}\mathcal{N}_{a,\ b,\ k} < \infty, \tag{4.25}$$

$$\mathcal{M}_a := \max_{K+1\le k\le N}\mathcal{M}_{a,\ k} < \infty \tag{4.26}$$

and

$$\mathcal{Q} := \max\{\mathcal{N}_{a,\ b},\ \mathcal{M}_a\}. \tag{4.27}$$

Our final technical proposition is as follows.

Lemma A2. *Let $N \geq 5,\ K \geq 2,\ 1 \leq k \leq N,\ 1 \leq l \leq K-1,\ K+1 \leq r \leq q-1,\ r+1 \leq q \leq N-1$, the integral kernels $G_k(x): I \to \mathbb{R},\ G_k(x) \in C(I)$ and $G_k(0) = G_k(2\pi)$ for all $1 \leq k \leq N$.*

a) Let $a_k > 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ for $1 \leq k \leq l$. Then $\mathcal{N}_{a,\ b,\ k} < \infty$.

b) Let $a_k = 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ for $l+1 \leq k \leq K$. Then $\mathcal{N}_{0,\ b,\ k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(I)} = 0. \tag{4.28}$$

c) Let $a_k > 0$ and $a_k \neq n^2,\ n \in \mathbb{Z}$ for $K+1 \leq k \leq r$. Then $\mathcal{M}_{a,\ k} < \infty$.

d) Let $a_k = n_k^2,\ n_k \in \mathbb{N}$ for $r+1 \leq k \leq q$. Then $\mathcal{M}_{a,\ k} < \infty$ if and only if

$$\left(G_k(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\right)_{L^2(I)} = 0 \tag{4.29}$$

holds.

e) Let $a_k = 0$ for $q+1 \leq k \leq N$. Then $\mathcal{M}_{0,\ k} < \infty$ if and only if orthogonality condition (4.28) is valid.

Proof. In both cases a) and b) of the lemma $\dfrac{G_{k,n}}{n^2 - a_k - i b_k n} \in l^\infty$ yields $\dfrac{n^2 G_{k,n}}{n^2 - a_k - i b_k n} \in l^\infty$. Indeed, $\dfrac{n^2 G_{k,n}}{n^2 - a_k - i b_k n}$ can be written as

$$G_{k,n} + a_k \frac{G_{k,n}}{n^2 - a_k - i b_k n} + i b_k \frac{n G_{k,n}}{n^2 - a_k - i b_k n}. \tag{4.30}$$

The first term in (4.30) is bounded by means of (4.22) under the given conditions. The third term in (4.30) can be easily estimated from above in the absolute value using (4.22) as well as

$$|b_k| \frac{|n||G_{k,n}|}{\sqrt{(n^2 - a_k)^2 + b_k^2 n^2}} \leq |G_{k,n}| \leq \sqrt{2\pi} \|G_k(x)\|_{C(I)} < \infty.$$

Thus, $\dfrac{G_{k,n}}{n^2 - a_k - i b_k n} \in l^\infty$ implies the boundedness of $\dfrac{n^2 G_{k,n}}{n^2 - a_k - i b_k n}$.
To treat the case a) of the lemma, we need to consider

$$\frac{|G_{k,n}|}{\sqrt{(n^2 - a_k)^2 + b_k^2 n^2}}. \tag{4.31}$$

Obviously, the denominator in fraction (4.31) can be estimated from below by a positive constant. Let us apply (4.22) to the numerator in (4.31). Hence $\mathcal{N}_{a,\ b,\ k} < \infty$ in the situation a) of our lemma when $a_k > 0$.

Let us establish the validity of the result of the part b) when a_k vanishes. Evidently, $\frac{G_{k,n}}{n^2 - ib_k n} \in l^\infty$ if and only if $G_{k,0} = 0$. This is equivalent to orthogonality relation (4.28).

In this case we easily obtain the upper bound

$$\left|\frac{G_{k,n}}{n^2 - ib_k n}\right| = \frac{|G_{k,n}|}{|n|\sqrt{n^2 + b_k^2}} \le \sqrt{2\pi}\frac{\|G_k(x)\|_{C(I)}}{|b_k|} < \infty$$

using (4.22) along with our assumptions.

In the cases c), d) and e) of the lemma we can express

$$\frac{n^2 G_{k,n}}{n^2 - a_k} = G_{k,n} + a_k \frac{G_{k,n}}{n^2 - a_k}. \tag{4.32}$$

By virtue of (4.22), the boundedness of $\frac{G_{k,n}}{n^2 - a_k}$ yields $\frac{n^2 G_{k,n}}{n^2 - a_k} \in l^\infty$.

In the situation c) we have $\frac{G_{k,n}}{n^2 - a_k} \in l^\infty$ since such expression does not contain any singularities and the result of our lemma is obvious.

In the part d) the quantity $\frac{G_{k,n}}{n^2 - n_k^2} \in l^\infty$ if and only if $G_{k,\pm n_k} = 0$. This is equivalent to orthogonality conditions (4.29).

In the case e) the expression $\frac{G_{k,n}}{n^2} \in l^\infty$ if and only if $G_{k,0} = 0$, which is equivalent to orthogonality relation (4.28). ■

Acknowledgement

V. V. is grateful to Israel Michael Sigal for the partial support by the NSERC grant NA 7901.

References

[1] G.L. Alfimov, A.S. Korobeinikov, C.J. Lustri, D.E. Pelinovsky. *Standing lattice solitons in the discrete NLS equation with saturation,* Nonlinearity, **32** (2019), no. 9, 3445–3484.

[2] M.S. Agranovich. *Elliptic boundary problems,* Encyclopaedia Math. Sci., **79**, Partial Differential Equations, IX, Springer, Berlin (1997), 1–144.

[3] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter. *Spatial structures and generalized travelling waves for an integro- differential equation,* Discrete Contin. Dyn. Syst. Ser. B, **13** (2010), no. 3, 537–557.

[4] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik. *The non-local Fisher-KPP equation: travelling waves and steady states,* Nonlinearity, **22** (2009), no. 12, 2813–2844.

[5] H. Berestycki, F. Hamel, N. Nadirashvili. *The speed of propagation for KPP type problems. I: Periodic framework,* J. Eur. Math. Soc. (JEMS), **7** (2005), no. 2, 173–213.

[6] H. Brezis, L. Oswald. *Remarks on sublinear elliptic equations,* Nonlinear Anal., **10** (1986), no. 1, 55–64.

[7] A. Ducrot, M. Marion, V. Volpert. *Systémes de réaction-diffusion sans propriété de Fredholm*, C. R. Math. Acad. Sci. Paris, **340** (2005), no. 9, 659–664.

[8] A. Ducrot, M. Marion, V. Volpert. *Reaction-diffusion problems with non-Fredholm operators,* Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[9] M.A. Efendiev. *Fredholm structures, topological invariants and applications.* AIMS Series on Differential Equations & Dynamical Systems, **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO (2009), 205 pp.

[10] M.A. Efendiev. *Finite and infinite dimensional attractors for evolution equations of mathematical physics.* GAKUTO International Series. Mathematical Sciences and Applications, **33**. *Gakkōtosho Co., Ltd., Tokyo* (2010), 239 pp.

[11] M.A. Efendiev, L.A. Peletier. *On the large time behavior of solutions of fourth order parabolic equations and ε-entropy of their attractors*, C. R. Math. Acad. Sci. Paris, **344** (2007), no. 2, 93–96.

[12] M.A. Efendiev, V. Vougalter. *Solvability of some integro-differential equations with drift,* Osaka J. Math., **57** (2020), no. 2, 247–265.

[13] M.A. Efendiev, V. Vougalter. *Linear and nonlinear non-Fredholm operators and their applications,* Electron. Res. Arch., **30** (2022), no. 2, 515–534.

[14] H.G. Gebran, C.A. Stuart. *Fredholm and properness properties of quasilinear elliptic systems of second order,* Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91–124.

[15] H.G. Gebran, C.A. Stuart. *Exponential decay and Fredholm properties in second-order quasilinear elliptic systems,* J. Differential Equations, **249** (2010), no. 1, 94–117.

[16] S. Genieys, V. Volpert, P. Auger. *Pattern and waves for a model in population dynamics with nonlocal consumption of resources,* Math. Model. Nat. Phenom., **1** (2006), no. 1, 65–82.

[17] P.D. Hislop, I.M. Sigal. *Introduction to spectral theory. With applications to Schrödinger operators.* Applied Mathematical Sciences, **113**. Springer-Verlag, New York, (1996), 337 pp.

[18] M.A. Krasnosel'skii. *Topological methods in the theory of nonlinear integral equations.* International Series of Monographs on Pure and Applied Mathematics. Pergamon Press, **XI**, (1964), 395 pp.

[19] J. -L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications.* Vol. 1. (French) Travaux et Recherches Mathematiques, No. 17. Dunod, Paris (1968), 372 pp.

[20] P.J. Rabier, C.A. Stuart. *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$*,* Math. Nachr., **231** (2001), 129–168.

[21] L.R. Volevich. *Solubility of boundary value problems for general elliptic systems.* (Russian) Mat. Sb. (N.S.) **68** (110) (1965), 373–416; English translation: Amer. Math. Soc. Transl., **67** (1968), Ser. 2, 182–225.

[22] V. Volpert. *Elliptic partial differential equations.* Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monographs in Mathematics, **101**. Birkhäuser/Springer Basel AG, Basel (2011), 639 pp.

[23] V. Volpert, B. Kazmierczak, M. Massot, Z. Peradzynski. *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math. (Warsaw), **29** (2002), no. 2, 219–238.

[24] V. Volpert, V. Vougalter. *Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation.* Dispersal, individual movement and spatial ecology, Lecture Notes in Math., **2071**, Springer, Heidelberg (2013), 331–353.

[25] V. Vougalter, V. Volpert. *Solvability conditions for some non-Fredholm operators,* Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[26] V. Vougalter, V. Volpert. *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580.

[27] V. Vougalter, V. Volpert. *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), no. 1, 365–373.

[28] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), no. 4, 473–496.