RESEARCH ARTICLE

On solvability of some systems of Fredholm integro-differential equations with mixed diffusion in a square

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Dedicated to Dr. Alexander J. Zaslavski on the occasion of his 65th birthday

Abstract

We establish the existence in the sense of sequences of solutions for a certain system of integro-differential equations in a square in two dimensions with periodic boundary conditions involving the normal diffusion in one direction and the superdiffusion in the other direction in a constrained subspace of H^2 for the vector functions via the fixed point technique. The system of elliptic equations contains a second order differential operator, which satisfies the Fredholm property. It is demonstrated that, under certain reasonable technical conditions, the convergence in the appropriate function spaces of the integral kernels implies the existence and convergence in $H_c^2(\Omega, \mathbb{R}^N)$ of the solutions. We generalize our results derived in Efendiev and Vougalter (J Dyn Differ Equ, 2022. https://doi.org/10.1007/s10884-022-10199-2) for an analogous system studied in the whole $\mathbb{R}^2$ which involved non-Fredholm operators. Let us emphasize that the study of systems is more complicated than the scalar case.

Keywords Solvability conditions · Fredholm operators · Integro-differential systems · Mixed diffusion

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1 Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. Such properties of Fredholm operators are widely used in linear and nonlinear analysis.

Elliptic problems considered in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, the proper ellipticity and the Shapiro–Lopatinskii conditions are fulfilled (see e.g. [2, 9, 25, 29]). This is the main result of the theory of linear elliptic equations. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L: C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L: H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

For linear elliptic problems in unbounded domains the Fredholm property is satisfied if and only if, in addition to the conditions given above, limiting operators are invertible (see [30]). In certain simple cases, limiting operators can be constructed explicitly. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinities

$$a_\pm = \lim_{x \to \pm\infty} a(x), \quad b_\pm = \lim_{x \to \pm\infty} b(x), \quad c_\pm = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since coefficients here are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not have the Fredholm property, can be found explicitly via the standard Fourier transform, such that

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Limiting operators are invertible if and only if the essential spectrum does not contain the origin.

For general elliptic equations analogous assertions are valid. The Fredholm property is satisfied if the origin does not belong to the essential spectrum or if limiting operators are invertible. However, such conditions may not be written explicitly.

For non-Fredholm operators the usual solvability relations may not be applicable and in a general situation solvability conditions are not known. But there are some

classes of operators for which solvability relations were derived recently. Let us illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$, where a is a positive constant. The operator L here coincides with its limiting operators. The corresponding homogeneous equation has a nonzero bounded solution such that the Fredholm property is not satisfied. However, since the operator in (1.1) has constant coefficients, we can apply the standard Fourier transform to obtain the solution explicitly. The solvability conditions can be formulated as follows. If $f(x) \in L^2(\mathbb{R}^d)$ and $xf(x) \in L^1(\mathbb{R}^d)$, then there exists a unique solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{d/2}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \text{ a.e.}$$

(see [35, Lemmas 5 and 6]). Here $S^d_{\sqrt{a}}$ stands for the sphere in $\mathbb{R}^d$ of radius $\sqrt{a}$ centered at the origin. Thus, despite the fact that the Fredholm property is not satisfied for the operator, the solvability conditions are formulated analogously. Clearly, such similarity is only formal since the range of the operator is not closed.

In the case of the operator involving a potential function,

$$Lu \equiv \Delta u + a(x)u = f,$$

the standard Fourier transform is not applicable directly. Nevertheless, solvability relations in ℝ³ can be derived by a rather sophisticated application of the theory of the self-adjoint Schrödinger type operators (see [33]). Similarly to the constant coefficient case, solvability conditions are written in terms of orthogonality to the solutions of the adjoint homogeneous problem. There are several other examples of linear elliptic non-Fredholm operators for which solvability relations can be obtained (see [13, 30–32, 35]).

Solvability relations play a crucial role in the analysis of nonlinear elliptic problems. When the operators without the Fredholm property are involved, in spite of certain progress in understanding of linear equations, there exist only a few examples where nonlinear non-Fredholm operators were analyzed (see [7, 8, 12–16, 34–37]). Fredholm structures, topological invariants and their applications were considered in [9]. The article [10] is devoted to finite- and infinite-dimensional attractors for evolution equations of mathematical physics. The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [11]. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was investigated in [18]. The works [21, 27] are crucial for understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in second-order quasilinear elliptic systems were discussed in [22]. A local bifurcation theorem for C^1-Fredholm

maps was established in [19]. In [20] the authors developed a degree theory for C^2-Fredholm mappings of zero index between Banach spaces. Standing lattice solitons in the discrete NLS equation with saturation were covered in [1]. The present work deals with another class of the stationary nonlinear systems of equations, for which the Fredholm property is satisfied:

$$\frac{\partial^2 u_k}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}}\, u_k + \int_\Omega G_k(x-y) F_k(u_1(y), u_2(y), \ldots, u_N(y), y)\, dy = 0 \quad (1.2)$$

with $1 \leqslant k \leqslant N$, $N \geqslant 2$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$ and the square $\Omega := [0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions specified further. Here and below the vector function

$$u := (u_1, u_2, \ldots, u_N)^T \in \mathbb{R}^N. \quad (1.3)$$

We generalize the results obtained for an analogous system in the whole $\mathbb{R}^2$ studied in [16]. It involved the non-Fredholm operators. For the solvability of single equations of this kind, see [12, 17]. The novelty of such works is that in the diffusion terms we add the free Laplacian in the x_1 variable to the negative Laplace operator in x_2 raised to a fractional power $0 < s_k < 1$, $1 \leqslant k \leqslant N$, $N \geqslant 2$ and defined via the spectral calculus. As distinct from the analogous system of equations discussed in [16], in the present article we restrict our attention to $s_k = 1/2$ for all k. The models of this type are new. They are not well understood, especially in the context of nonlocal reaction-diffusion equations. The difficulty we have to overcome is that such problems become anisotropic and it is more complicated to derive the desired estimates when working with them. In the population dynamics in Mathematical Biology integro-differential equations describe the models with the intra-specific competition and nonlocal consumption of resources (see e.g. [3, 4]). It is crucial to consider the problems of this type from the point of view of understanding of the spread of the viral infections, since many countries have to deal with pandemias. We use the explicit form of solvability conditions and establish the existence of solutions of our nonlinear system. In the case of the standard Laplacian in the diffusion terms, the system of equations analogous to (1.2) was covered in [37] (see also [34]) in the whole space and on a finite interval with periodic boundary conditions. The solvability of the integro-differential problems involving in the diffusion terms only the negative Laplacian raised to a fractional power was actively studied in recent years in the context of the anomalous diffusion (see e.g. [15, 36]). The anomalous diffusion can be described as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the Laplacian (see [26]). In [28] the authors discuss the mixed local-nonlocal semilinear elliptic problems driven by the superposition of Brownian and Levy processes and establish the L^∞-boundedness of any weak solution. The work [6] deals with a new type of mixed local and nonlocal equations under the Neumann conditions. The

spectral properties associated to a weighted eigenvalue problem are considered and a global estimate for subsolutions is presented.

2 Formulation of results

Technical assumptions in the present work will be analoguos to the ones of [17], adapted to the work with vector functions. Performing analysis in the Sobolev spaces for vector functions is more difficult. For the nonlinear part of system of equations (1.2) the following regularity conditions will be assumed. Here $x = (x_1, x_2) \in \Omega$.

Assumption 2.1 Let $1 \leqslant k \leqslant N$. Let functions $F_k(u, x) \colon \mathbb{R}^N \times \Omega \to \mathbb{R}$ satisfy the Carathéodory condition (see [24]):

$$\sqrt{\sum_{k=1}^{N} F_k^2(u, x)} \leqslant K |u|_{\mathbb{R}^N} + h(x) \quad \text{for } u \in \mathbb{R}^N, \ x \in \Omega, \tag{2.1}$$

with a constant $K > 0$ and $h(x) \colon \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Moreover, let they be Lipschitz continuous functions such that for any $u^{(1),(2)} \in \mathbb{R}^N$, $x \in \Omega$:

$$\sqrt{\sum_{k=1}^{N} \left(F_k(u^{(1)}, x) - F_k(u^{(2)}, x) \right)^2} \leqslant L \left| u^{(1)} - u^{(2)} \right|_{\mathbb{R}^N} \tag{2.2}$$

with a constant $L > 0$. Furthermore, for $1 \leqslant k \leqslant N$,

$$F_k(u, 0, x_2) = F_k(u, 2\pi, x_2) \quad \text{for } u \in \mathbb{R}^N, \ 0 \leqslant x_2 \leqslant 2\pi,$$

and

$$F_k(u, x_1, 0) = F_k(u, x_1, 2\pi) \quad \text{for } u \in \mathbb{R}^N, \ 0 \leqslant x_1 \leqslant 2\pi.$$

Here and further the norm of a vector function given by (1.3) is

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was covered in [5]. The nonlinear function there was allowed to have a sublinear growth. To demonstrate the existence of solutions of (1.2), we will use an auxiliary system with $1 \leqslant k \leqslant N$, $N \geqslant 2$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$, namely

$$-\frac{\partial^2 u_k}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} \, u_k = \int_{\Omega} G_k(x - y) F_k(v_1(y), v_2(y), \ldots, v_N(y), y) \, dy. \tag{2.3}$$

We denote

$$(f_1(x_1, x_2), f_2(x_1, x_2))_{L^2(\Omega)} := \int_0^{2\pi} \int_0^{2\pi} f_1(x_1, x_2) \bar{f}_2(x_1, x_2) \, dx_1 dx_2.$$

Let us use the Sobolev space

$$H^2(\Omega) := \left\{ \phi(x_1, x_2) : \Omega \to \mathbb{R} \,\middle|\, \begin{array}{l} \phi(x_1, x_2), \, \Delta\phi(x_1, x_2) \in L^2(\Omega), \, \phi(0, x_2) = \phi(2\pi, x_2), \\ \frac{\partial \phi}{\partial x_1}(0, x_2) = \frac{\partial \phi}{\partial x_1}(2\pi, x_2), \, 0 \leqslant x_2 \leqslant 2\pi, \\ \phi(x_1, 0) = \phi(x_1, 2\pi), \, \frac{\partial \phi}{\partial x_2}(x_1, 0) = \frac{\partial \phi}{\partial x_2}(x_1, 2\pi), \, 0 \leqslant x_1 \leqslant 2\pi \end{array} \right\}.$$

Here and further the cumulative Laplace operator is $\Delta := \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. We introduce the following auxiliary constrained subspace:

$$H_0^2(\Omega) := \{\phi(x_1, x_2) \in H^2(\Omega) \mid (\phi(x_1, x_2), 1)_{L^2(\Omega)} = 0\}. \tag{2.4}$$

Evidently, (2.4) is a Hilbert space as well (see e.g. [23, Chapter 2.1]). It is equipped with the norm

$$\|\phi\|_{H_0^2(\Omega)}^2 := \|\phi\|_{L^2(\Omega)}^2 + \|\Delta\phi\|_{L^2(\Omega)}^2.$$

The resulting space used to establish the existence of solutions $u(x) : \Omega \to \mathbb{R}^N$ of system (2.3) will be the direct sum of the spaces

$$H_c^2(\Omega, \mathbb{R}^N) := \bigoplus_{k=1}^N H_0^2(\Omega).$$

The corresponding Sobolev norm of a vector function is given by

$$\|u\|_{H_c^2(\Omega, \mathbb{R}^N)}^2 := \sum_{k=1}^N \left\{ \|u_k\|_{L^2(\Omega)}^2 + \|\Delta u_k\|_{L^2(\Omega)}^2 \right\}, \tag{2.5}$$

where $u(x) : \Omega \to \mathbb{R}^N$. Let us also use the norm

$$\|u\|_{L^2(\Omega, \mathbb{R}^N)}^2 := \sum_{k=1}^N \|u_k\|_{L^2(\Omega)}^2.$$

As seen from Assumption 2.1, we do not consider the higher powers of the nonlinearities than the first one. This is restrictive from the point of view of applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\Omega, \mathbb{R}^N)$ to $L^2(\Omega, \mathbb{R}^N)$. System of equations (2.3) involves the operator

$$L_r := -\frac{\partial^2}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} : \quad H_0^2(\Omega) \to L^2(\Omega). \tag{2.6}$$

Its eigenvalues are given by

$$\lambda_{r,n_1,n_2} := n_1^2 + |n_2|, \quad (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0).$$

The corresponding eigenfunctions are

$$\frac{e^{in_1 x_1}}{\sqrt{2\pi}} \frac{e^{in_2 x_2}}{\sqrt{2\pi}}, \quad (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \backslash (0, 0).$$

Clearly, (2.6) is a Fredholm operator and its kernel is trivial. In the present work we will establish that under certain reasonable technical assumptions system (2.3) defines a map $T_r \colon H_c^2(\Omega, \mathbb{R}^N) \to H_c^2(\Omega, \mathbb{R}^N)$, which is a strict contraction.

Theorem 2.2 *Let $N \geqslant 2$, $1 \leqslant k \leqslant N$, Assumption 2.1 hold, the functions $G_k(x_1, x_2) \colon \Omega \to \mathbb{R}$ be such that $G_k(0, x_2) = G_k(2\pi, x_2)$ with $0 \leqslant x_2 \leqslant 2\pi$ and $G_k(x_1, 0) = G_k(x_1, 2\pi)$ with $0 \leqslant x_1 \leqslant 2\pi$. Furthermore, let $G_k(x_1, x_2) \in C(\Omega)$ and $\partial G_k(x_1, x_2)/\partial x_2 \in L^1(\Omega)$. We also assume that orthogonality conditions (4.7) hold for $1 \leqslant k \leqslant N$ and that $2\sqrt{2}\pi \mathcal{N}_r L < 1$.*

Then the map $T_r v = u$ on $H_c^2(\Omega, \mathbb{R}^N)$ defined by system (2.3) has a unique fixed point v_r, which is the only solution of problem (1.2) in $H_c^2(\Omega, \mathbb{R}^N)$. This fixed point v_r is nontrivial provided the Fourier coefficients $G_{k,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for a certain $1 \leqslant k \leqslant N$ and some $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Related to system of equations (1.2) in the square Ω, we study the following sequence of approximate systems:

$$\frac{\partial^2 u_k^{(m)}}{\partial x_1^2} - \sqrt{-\frac{\partial^2}{\partial x_2^2}} u_k^{(m)} + \int_\Omega G_{k,m}(x - y) F_k\big(u_1^{(m)}(y), u_2^{(m)}(y), \ldots, u_N^{(m)}(y), y\big) \, dy = 0, \quad (2.7)$$

with $1 \leqslant k \leqslant N$, $N \geqslant 2$, $m \in \mathbb{N}$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$. Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^\infty$ tends to $G_k(x)$ as $m \to \infty$ in the function spaces listed below. We demonstrate that, under appropriate technical conditions, each of systems (2.7) has a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, limiting system of equations (1.2) admits a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$. This is the so-called *existence of solutions in the sense of sequences*. In this case, solvability relations can be formulated for the iterated kernels $G_{k,m}$. They yield the convergence of kernels in terms of the Fourier transforms (see Appendix) and, as a consequence, the convergence of solutions (Theorem 2.3). Similar ideas in the context of the standard Schrödinger type operators were exploited in [13, 32]. Our second main result is as follows.

Theorem 2.3 *Let $m \in \mathbb{N}$, $N \geqslant 2$, $1 \leqslant k \leqslant N$, Assumption 2.1 hold, and the functions $G_{k,m}(x_1, x_2) \colon \Omega \to \mathbb{R}$ be such that $G_{k,m}(0, x_2) = G_{k,m}(2\pi, x_2)$ with $0 \leqslant x_2 \leqslant 2\pi$ and $G_{k,m}(x_1, 0) = G_{k,m}(x_1, 2\pi)$ with $0 \leqslant x_1 \leqslant 2\pi$. Moreover, let*

$$G_{k,m}(x_1, x_2) \in C(\Omega), \quad G_{k,m}(x_1, x_2) \to G_k(x_1, x_2) \text{ in } C(\Omega) \text{ as } m \to \infty.$$

In addition to that, let

$$\frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \to \frac{\partial G_k(x_1, x_2)}{\partial x_2} \text{ in } L^1(\Omega) \text{ as } m \to \infty.$$

We also assume that for each $1 \leqslant k \leqslant N$, $m \in \mathbb{N}$ orthogonality condition (4.11) is valid. Finally, we suppose that (4.12) holds for each $m \in \mathbb{N}$ with a certain fixed $0 < \varepsilon < 1$.

Then each system of equations (2.7) admits a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, limiting system (1.2) has a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$ such that $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system of equations (2.7) does not vanish identically in Ω provided the Fourier coefficients $G_{k,m,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for a certain $1 \leqslant k \leqslant N$ and some pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Analogously, the unique solution $u(x)$ of limiting system (1.2) is nontrivial in Ω if $G_{k,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for some $1 \leqslant k \leqslant N$ and a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

Remark 2.4 In this article we deal with the real-valued vector functions under the conditions imposed on $F_k(u, x_1, x_2)$, $G_{k,m}(x_1, x_2)$ and $G_k(x_1, x_2)$ involved in the integral terms of the approximate and limiting systems of equations discussed here.

Remark 2.5 The significance of Theorem 2.3 is the continuous dependence of solutions with respect to the integral kernels.

3 Proofs of main results

Proof of Theorem 2.2 Let us first suppose that for a certain $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$ of system of equations (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$ will satisfy the homogeneous system

$$-\frac{\partial^2 w_k}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} \, w_k = 0, \quad 1 \leqslant k \leqslant N.$$

Clearly, the operator $L_r \colon H_0^2(\Omega) \to L^2(\Omega)$ defined in (2.6) does not have any nontrivial zero modes. Thus, the vector function $w(x)$ vanishes identically in the square Ω.

We choose an arbitrarily $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$. Let us apply the Fourier transform (4.1) to both sides of system (2.3). This gives us for $1 \leqslant k \leqslant N$, $N \geqslant 2$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ that

$$\begin{aligned} u_{k,n_1,n_2} &= 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}}{n_1^2 + |n_2|}, \\ (n_1^2 + n_2^2) u_{k,n_1,n_2} &= 2\pi \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}}{n_1^2 + |n_2|}. \end{aligned} \tag{3.1}$$

Here $f_{k,n_1,n_2} := F_k(v(x), x)_{n_1,n_2}$. Evidently, from above we have the estimates

$$|u_{k,n_1,n_2}| \leqslant 2\pi \mathcal{N}_{r,k}|f_{k,n_1,n_2}|, \quad \left|(n_1^2 + n_2^2) u_{k,n_1,n_2}\right| \leqslant 2\pi \mathcal{N}_{r,k}|f_{k,n_1,n_2}|$$

with $1 \leqslant k \leqslant N$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Obviously, all $\mathcal{N}_{r,k} < \infty$ due to Lemma 4.1 from Appendix under the given conditions. This allows us to obtain the upper bound on the norm as

$$
\begin{aligned}
\|u\|_{H_c^2(\Omega, \mathbb{R}^N)}^2 &= \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |u_{k,n_1,n_2}|^2 + \sum_{k=1}^{N} \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} \left|(n_1^2 + n_2^2) u_{k,n_1,n_2}\right|^2 \\
&\leqslant 8\pi^2 \sum_{k=1}^{N} \mathcal{N}_{r,k}^2 \|F_k(v(x), x)\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{3.2}
$$

Let us recall inequality (2.1) of Assumption 2.1. Hence, the right side of (3.2) is finite for $v(x) \in L^2(\Omega, \mathbb{R}^N)$. Thus, for any $v(x) \in H_c^2(\Omega, \mathbb{R}^N)$ there exists a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$ of system (2.3) such that its Fourier image is given by (3.1). Therefore, the map $T_r: H_c^2(\Omega, \mathbb{R}^N) \to H_c^2(\Omega, \mathbb{R}^N)$ is well defined. This allows us to choose arbitrarily the vector functions $v^{(1),(2)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$ such that their images $u^{(1),(2)} := T_r v^{(1),(2)} \in H_c^2(\Omega, \mathbb{R}^N)$. By means of (2.3), we have for $1 \leqslant k \leqslant N$, $N \geqslant 2$, $x = (x_1, x_2) \in \Omega$, $y = (y_1, y_2) \in \Omega$,

$$-\frac{\partial^2 u_k^{(1)}}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} u_k^{(1)} = \int_\Omega G_k(x - y) F_k\left(v_1^{(1)}(y), v_2^{(1)}(y), \ldots, v_N^{(1)}(y), y\right) dy, \tag{3.3}$$

$$-\frac{\partial^2 u_k^{(2)}}{\partial x_1^2} + \sqrt{-\frac{\partial^2}{\partial x_2^2}} u_k^{(2)} = \int_\Omega G_k(x - y) F_k\left(v_1^{(2)}(y), v_2^{(2)}(y), \ldots, v_N^{(2)}(y), y\right) dy. \tag{3.4}$$

Let us apply the Fourier transform (4.1) to both sides of the equations of systems (3.3), (3.4). This gives us for $1 \leqslant k \leqslant N$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$,

$$u_{k,n_1,n_2}^{(1)} = 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}^{(1)}}{n_1^2 + |n_2|}, \quad u_{k,n_1,n_2}^{(2)} = 2\pi \frac{G_{k,n_1,n_2} f_{k,n_1,n_2}^{(2)}}{n_1^2 + |n_2|}, \tag{3.5}$$

$$(n_1^2 + n_2^2) u_{k,n_1,n_2}^{(1)} = 2\pi \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}^{(1)}}{n_1^2 + |n_2|}, \tag{3.6}$$

$$(n_1^2 + n_2^2) u_{k,n_1,n_2}^{(2)} = 2\pi \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2} f_{k,n_1,n_2}^{(2)}}{n_1^2 + |n_2|}. \tag{3.7}$$

Here $f_{k,n_1,n_2}^{(1)}$ and $f_{k,n_1,n_2}^{(2)}$ stand for the images of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively under transform (4.1). By virtue of (3.5), (3.6) and (3.7), we derive the inequalities

$$\left|u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}\right| \leqslant 2\pi \mathcal{N}_{r,k}\left|f_{k,n_1,n_2}^{(1)} - f_{k,n_1,n_2}^{(2)}\right|,$$

$$\left|(n_1^2 + n_2^2)\left[u_{k,n_1,n_2}^{(1)} - u_{k,n_1,n_2}^{(2)}\right]\right| \leqslant 2\pi \mathcal{N}_{r,k}\left|f_{k,n_1,n_2}^{(1)} - f_{k,n_1,n_2}^{(2)}\right|$$

with $1 \leqslant k \leqslant N$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Hence,

$$
\begin{aligned}
\big\| u^{(1)} &- u^{(2)} \big\|_{H_c^2(\Omega, \mathbb{R}^N)}^2 \\
&= \sum_{k=1}^{N} \sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} \big| u_{k, n_1, n_2}^{(1)} - u_{k, n_1, n_2}^{(2)} \big|^2 \\
&\qquad + \sum_{k=1}^{N} \sum_{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}} \big| (n_1^2 + n_2^2) \big[u_{k, n_1, n_2}^{(1)} - u_{k, n_1, n_2}^{(2)} \big] \big|^2 \\
&\leqslant 8\pi^2 \mathcal{N}_r^2 \sum_{k=1}^{N} \big\| F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x) \big\|_{L^2(\Omega)}^2,
\end{aligned}
$$

with $\mathcal{N}_r$ defined in (4.6). We recall condition (2.2) of Assumption 2.1. Thus,

$$
\big\| T_r v^{(1)} - T_r v^{(2)} \big\|_{H_c^2(\Omega, \mathbb{R}^N)} \leqslant 2\sqrt{2}\,\pi \mathcal{N}_r L \big\| v^{(1)} - v^{(2)} \big\|_{H_c^2(\Omega, \mathbb{R}^N)}. \tag{3.8}
$$

The constant in the right side of (3.8) is less than one as we assumed. Therefore, by the Fixed Point Theorem, there exists a unique vector function $v_r \in H_c^2(\Omega, \mathbb{R}^N)$ such that $T_r v_r = v_r$. This is the only solution of system of equations (1.2) in $H_c^2(\Omega, \mathbb{R}^N)$. Let us suppose that $v_r(x)$ vanishes identically in Ω. This will contradict to the given condition that the Fourier coefficients $G_{k, n_1, n_2} F_k(0, x)_{n_1, n_2} \neq 0$ for some $1 \leqslant k \leqslant N$ and a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.   □

Let us proceed to establishing the solvability in the sense of sequences for our system of integro-differential equations in the square Ω.

Proof of Theorem 2.3 By Theorem 2.2, each system (2.7) possesses a unique solution $u^{(m)}(x) \in H_c^2(\Omega, \mathbb{R}^N)$, $m \in \mathbb{N}$. Limiting system of equations (1.2) admits a unique solution $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$ by Lemma 4.2 along with Theorem 2.2. Let us apply the Fourier transform (4.1) to both sides of systems (1.2) and (2.7). Hence, for $1 \leqslant k \leqslant N$, $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$ and $m \in \mathbb{N}$, we obtain

$$
\begin{aligned}
u_{k, n_1, n_2} &= 2\pi \, \frac{G_{k, n_1, n_2} \varphi_{k, n_1, n_2}}{n_1^2 + |n_2|}, \\
(n_1^2 + n_2^2) u_{k, n_1, n_2} &= 2\pi \, \frac{(n_1^2 + n_2^2) G_{k, n_1, n_2} \varphi_{k, n_1, n_2}}{n_1^2 + |n_2|}, \\
u_{k, n_1, n_2}^{(m)} &= 2\pi \, \frac{G_{k, m, n_1, n_2} \varphi_{k, n_1, n_2}^{(m)}}{n_1^2 + |n_2|}, \\
(n_1^2 + n_2^2) u_{k, n_1, n_2}^{(m)} &= 2\pi \, \frac{(n_1^2 + n_2^2) G_{k, m, n_1, n_2} \varphi_{k, n_1, n_2}^{(m)}}{n_1^2 + |n_2|}.
\end{aligned} \tag{3.9}
$$

In formulas (3.9), φ_{k,n_1,n_2} and $\varphi_{k,n_1,n_2}^{(m)}$ stand for the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively under transform (4.1). Clearly,

$$\left| u_{k,n_1,n_2}^{(m)} - u_{k,n_1,n_2} \right| \leqslant 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |\varphi_{k,n_1,n_2}|$$
$$+ 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \left| \varphi_{k,n_1,n_2}^{(m)} - \varphi_{k,n_1,n_2} \right|$$

so that

$$\left\| u_k^{(m)} - u_k \right\|_{L^2(\Omega)} \leqslant 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \| F_k(u(x), x) \|_{L^2(\Omega)}$$
$$+ 2\pi \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \left\| F_k(u^{(m)}(x), x) - F_k(u(x), x) \right\|_{L^2(\Omega)}.$$

We recall bound (2.2) of Assumption 2.1. Hence,

$$\sqrt{\sum_{k=1}^N \left\| F_k(u^{(m)}(x), x) - F_k(u(x), x) \right\|_{L^2(\Omega)}^2} \leqslant L \| u^{(m)}(x) - u(x) \|_{L^2(\Omega, \mathbb{R}^N)}. \tag{3.10}$$

Thus, using (4.9) and (4.10), we derive

$$\left\| u^{(m)}(x) - u(x) \right\|_{L^2(\Omega, \mathbb{R}^N)}^2$$
$$\leqslant 8\pi^2 \sum_{k=1}^N \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}^2 \| F_k(u(x), x) \|_{L^2(\Omega)}^2$$
$$+ 8\pi^2 [\mathcal{N}_r^{(m)}]^2 L^2 \left\| u^{(m)}(x) - u(x) \right\|_{L^2(\Omega, \mathbb{R}^N)}^2.$$

By (4.12), we obtain

$$\left\| u^{(m)}(x) - u(x) \right\|_{L^2(\Omega, \mathbb{R}^N)}^2$$
$$\leqslant \frac{8\pi^2}{\varepsilon(2 - \varepsilon)} \sum_{k=1}^N \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}^2 \| F_k(u(x), x) \|_{L^2(\Omega)}^2.$$

Inequality (2.1) of Assumption 2.1 yields that all $F_k(u(x), x) \in L^2(\Omega)$ for $u(x) \in H_c^2(\Omega, \mathbb{R}^N)$. Hence, under the given conditions

$$u^{(m)}(x) \to u(x), \quad m \to \infty, \tag{3.11}$$

in $L^2(\Omega, \mathbb{R}^N)$ by Lemma 4.2 of Appendix. Formulas (3.9) yield

$$\left| \left(n_1^2 + n_2^2 \right) u_{k,n_1,n_2}^{(m)} - \left(n_1^2 + n_2^2 \right) u_{k,n_1,n_2} \right|$$

$$\leqslant 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |\varphi_{k,n_1,n_2}|$$

$$+ 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} |\varphi_{k,n_1,n_2}^{(m)} - \varphi_{k,n_1,n_2}|$$

so that

$$\left\| \Delta u_k^{(m)}(x) - \Delta u_k(x) \right\|_{L^2(\Omega)} \leqslant 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}$$

$$\times \left\| F_k(u(x), x) \right\|_{L^2(\Omega)}$$

$$+ 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}$$

$$\times \left\| F_k(u^{(m)}(x), x) - F(u(x), x) \right\|_{L^2(\Omega)}.$$

Inequality (3.10) allows us to derive the estimate

$$\left\| \Delta u_k^{(m)}(x) - \Delta u_k(x) \right\|_{L^2(\Omega)} \leqslant 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}$$

$$\times \left\| F_k(u(x), x) \right\|_{L^2(\Omega)}$$

$$+ 2\pi \left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}$$

$$\times L \| u^{(m)}(x) - u(x) \|_{L^2(\Omega, \mathbb{R}^N)}.$$

Let us recall Lemma 4.2 of Appendix along with statement (3.11). We obtain that $\Delta u_k^{(m)}(x) \to \Delta u_k(x)$ in $L^2(\Omega)$ as $m \to \infty$ for $1 \leqslant k \leqslant N$. By definition (2.5) of the norm, we have $u^{(m)}(x) \to u(x)$ in $H_c^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$.

Let us suppose the solution $u^{(m)}(x)$ of system (2.7) discussed above vanishes identically in the square Ω for some $m \in \mathbb{N}$. This will contradict to the posed condition that the Fourier coefficients $G_{k,m,n_1,n_2} F_k(0, x)_{n_1,n_2} \neq 0$ for some $1 \leqslant k \leqslant N$ and a certain pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. A similar argument is valid for the solution $u(x)$ of the limiting problem (1.2). $\square$

4 Appendix

Let the function $G_k(x_1, x_2): \Omega \to \mathbb{R}$ be such that $G_k(0, x_2) = G_k(2\pi, x_2)$ with $0 \leqslant x_2 \leqslant 2\pi$ and $G_k(x_1, 0) = G_k(x_1, 2\pi)$ with $0 \leqslant x_1 \leqslant 2\pi$. Its Fourier image on the square equals to

$$G_{k,n_1,n_2} := \int_0^{2\pi} \int_0^{2\pi} G_k(x_1, x_2) \frac{e^{-in_1 x_1}}{\sqrt{2\pi}} \frac{e^{-in_2 x_2}}{\sqrt{2\pi}} dx_1 dx_2, \quad (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}, \quad (4.1)$$

with

$$G_k(x_1, x_2) = \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} G_{k,n_1,n_2} \frac{e^{in_1 x_1}}{\sqrt{2\pi}} \frac{e^{in_2 x_2}}{\sqrt{2\pi}}, \quad (x_1, x_2) \in \Omega.$$

Obviously, the upper bound

$$\|G_{k,n_1,n_2}\|_{l^\infty} \leqslant \frac{1}{2\pi} \|G_k(x_1, x_2)\|_{L^1(\Omega)} \tag{4.2}$$

is valid with $\|G_{k,n_1,n_2}\|_{l^\infty} := \sup_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} |G_{k,n_1,n_2}|$. Evidently, (4.2) yields

$$\|n_2 G_{k,n_1,n_2}\|_{l^\infty} \leqslant \frac{1}{2\pi} \left\| \frac{\partial G_k(x_1, x_2)}{\partial x_2} \right\|_{L^1(\Omega)}. \tag{4.3}$$

Furthermore, for a function continuous in the square Ω, the inequality

$$\|G_k(x_1, x_2)\|_{L^1(\Omega)} \leqslant \|G_k(x_1, x_2)\|_{C(\Omega)} (2\pi)^2 \tag{4.4}$$

is valid with $\|G_k(x_1, x_2)\|_{C(\Omega)} := \max_{(x_1,x_2)\in\Omega} |G_k(x_1, x_2)|$. Let us introduce the following technical quantities:

$$\mathcal{N}_{r,k} := \max\left\{ \left\| \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \left\| \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \right\}, \tag{4.5}$$

with $1 \leqslant k \leqslant N$, $N \geqslant 2$. Under the assumptions of Lemma 4.1, all the expressions (4.5) will be finite. Thus,

$$\mathcal{N}_r := \max_{1\leqslant k\leqslant N} \mathcal{N}_{r,k} < \infty. \tag{4.6}$$

The auxiliary statements below are the adaptations of the ones used in [17] to study the single integro-differential problem with mixed diffusion, analogously to system of equations (1.2). Let us provide them for the convenience of the readers.

Lemma 4.1 *Let $N \geqslant 2$, $1 \leqslant k \leqslant N$, the functions $G_k(x_1, x_2): \Omega \to \mathbb{R}$ be such that $G_k(0, x_2) = G_k(2\pi, x_2)$ with $0 \leqslant x_2 \leqslant 2\pi$ and $G_k(x_1, 0) = G_k(x_1, 2\pi)$ with $0 \leqslant x_1 \leqslant 2\pi$. Moreover, let $G_k(x_1, x_2) \in C(\Omega)$ and $\partial G_k(x_1, x_2)/\partial x_2 \in L^1(\Omega)$. Then $\mathcal{N}_{r,k} < \infty$ if and only if*

$$(G_k(x_1, x_2), 1)_{L^2(\Omega)} = 0. \tag{4.7}$$

Proof Let us first demonstrate that under the given conditions

$$\frac{(n_1^2 + n_2^2)G_{k,n_1,n_2}}{n_1^2 + |n_2|} \in l^\infty.$$

Clearly, by (4.2) and (4.4),

$$\left| \frac{n_1^2 G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right| \leqslant \|G_{k,n_1,n_2}\|_{l^\infty} \leqslant 2\pi \|G_k(x_1, x_2)\|_{C(\Omega)} < \infty$$

as we assume. By (4.3), we have

$$\left| \frac{n_2^2 G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right| \leqslant \|n_2 G_{k,n_1,n_2}\|_{l^\infty} \leqslant \frac{1}{2\pi} \left\| \frac{\partial G_k(x_1, x_2)}{\partial x_2} \right\|_{L^1(\Omega)} < \infty$$

as assumed. Thus, $(n_1^2 + n_2^2)G_{k,n_1,n_2}/(n_1^2 + |n_2|)$ is bounded. We can write

$$\begin{aligned}
\frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} &= \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \chi_{\{(n_1,n_2) \in \mathbb{Z} \times \mathbb{Z} \,|\, n_1 = n_2 = 0\}} \\
&\quad + \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \chi_{\{(n_1,n_2) \in \mathbb{Z} \times \mathbb{Z} \,|\, n_1 = n_2 = 0\}^c}.
\end{aligned} \tag{4.8}$$

Here and further down χ_A will stand for the characteristic function of a set $A \subseteq \mathbb{Z} \times \mathbb{Z}$ and A^c will denote the complement of A. Obviously, the second term in the right side of (4.8) can be estimated from above in the absolute value by means of (4.2) along with (4.4) as

$$|G_{k,n_1,n_2}| \leqslant 2\pi \|G_k(x_1, x_2)\|_{C(\Omega)} < \infty$$

via one of our assumptions. Evidently, the first term in the right side of (4.8) is bounded if and only if $G_{k,0,0}$ vanishes. This is equivalent to orthogonality relation (4.7). □

Let us note that the proof of the lemma above uses only on a single orthogonality condition for each $1 \leqslant k \leqslant N$, $N \geqslant 2$, as distinct from the analogous case in the whole $\mathbb{R}^2$ considered in [16].

For the purpose of the studies of systems of equations (2.7), we will use the following auxiliary expressions:

$$\mathcal{N}_{r,k}^{(m)} := \max \left\{ \left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \left\| \frac{(n_1^2 + n_2^2)G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \right\}, \quad m \in \mathbb{N}, \quad (4.9)$$

where $1 \leqslant k \leqslant N$, $N \geqslant 2$. Under the conditions of Lemma 4.2, all expressions (4.9) will be finite. This will enable us to introduce

$$\mathcal{N}_r^{(m)} := \max_{1 \leqslant k \leqslant N} \mathcal{N}_{r,k}^{(m)}, \quad m \in \mathbb{N}. \tag{4.10}$$

The final statement of the article is as follows.

Lemma 4.2 *Let $m \in \mathbb{N}$, $N \geqslant 2$, $1 \leqslant k \leqslant N$, let the functions $G_{k,m}(x_1, x_2)\colon \Omega \to \mathbb{R}$ be such that $G_{k,m}(0, x_2) = G_{k,m}(2\pi, x_2)$ with $0 \leqslant x_2 \leqslant 2\pi$ and $G_{k,m}(x_1, 0) = G_{k,m}(x_1, 2\pi)$ with $0 \leqslant x_1 \leqslant 2\pi$. Furthermore, let*

$$G_{k,m}(x_1, x_2) \in C(\Omega), \quad G_{k,m}(x_1, x_2) \to G_k(x_1, x_2) \quad in \ C(\Omega) \ as \ m \to \infty.$$

Additionally,

$$\frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \in L^1(\Omega), \quad \frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} \to \frac{\partial G_k(x_1, x_2)}{\partial x_2} \ in \ L^1(\Omega) \ as \ m \to \infty.$$

We also suppose that for all $1 \leqslant k \leqslant N$, $m \in \mathbb{N}$,

$$(G_{k,m}(x_1, x_2), 1)_{L^2(\Omega)} = 0 \tag{4.11}$$

holds. Finally, we assume that

$$2\sqrt{2}\,\pi\,\mathcal{N}_r^{(m)} L \leqslant 1 - \varepsilon \tag{4.12}$$

is valid for each $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$.
    Then, for all $1 \leqslant k \leqslant N$, we have

$$\frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \to \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad m \to \infty, \tag{4.13}$$

$$\frac{(n_1^2 + n_2^2)\,G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \to \frac{(n_1^2 + n_2^2)\,G_{k,n_1,n_2}}{n_1^2 + |n_2|}, \quad m \to \infty \tag{4.14}$$

in l^∞ so that

$$\left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \to \left\| \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \quad m \to \infty, \tag{4.15}$$

$$\left\| \frac{(n_1^2 + n_2^2)\,G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \to \left\| \frac{(n_1^2 + n_2^2)\,G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}, \quad m \to \infty. \tag{4.16}$$

Moreover,

$$2\sqrt{2}\,\pi\,\mathcal{N}_r L \leqslant 1 - \varepsilon \tag{4.17}$$

holds.

Proof Evidently, under the stated assumptions all $\mathcal{N}_{r,k}^{(m)}$ are finite by Lemma 4.1, so $\mathcal{N}_r^{(m)} < \infty$, $m \in \mathbb{N}$.

It can be easily verified that the limiting kernels are periodic functions as well. Indeed, for $0 \leqslant x_2 \leqslant 2\pi$, we obtain

$$
\begin{aligned}
|G_k(0, x_2) - G_k(2\pi, x_2)| &\leqslant |G_{k,m}(0, x_2) - G_k(0, x_2)| \\
&\quad + |G_{k,m}(2\pi, x_2) - G_k(2\pi, x_2)| \\
&\leqslant 2\|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty,
\end{aligned}
$$

due to our assumptions. Hence,

$$
G_k(0, x_2) = G_k(2\pi, x_2) \quad \text{for} \ \ 0 \leqslant x_2 \leqslant 2\pi,
$$

with $1 \leqslant k \leqslant N$. Analogously, for $0 \leqslant x_1 \leqslant 2\pi$,

$$
\begin{aligned}
|G_k(x_1, 0) - G_k(x_1, 2\pi)| &\leqslant |G_{k,m}(x_1, 0) - G_k(x_1, 0)| \\
&\quad + |G_{k,m}(x_1, 2\pi) - G_k(x_1, 2\pi)| \\
&\leqslant 2\|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty,
\end{aligned}
$$

as we assumed. Thus,

$$
G_k(x_1, 0) = G_k(x_1, 2\pi) \quad \text{for} \ \ 0 \leqslant x_1 \leqslant 2\pi,
$$

where $1 \leqslant k \leqslant N$. Let us demonstrate that the limiting orthogonality relations

$$
(G_k(x_1, x_2), 1)_{L^2(\Omega)} = 0, \quad 1 \leqslant k \leqslant N, \tag{4.18}
$$

are valid. With the help of (4.11), we derive

$$
\begin{aligned}
|(G_k(x_1, x_2), 1)_{L^2(\Omega)}| &= \left| (G_k(x_1, x_2), 1)_{L^2(\Omega)} - (G_{k,m}(x_1, x_2), 1)_{L^2(\Omega)} \right| \\
&\leqslant \|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} (2\pi)^2 \to 0, \quad m \to \infty,
\end{aligned}
$$

via one of our assumptions, so (4.18) holds.

Hence, by Lemma 4.1, all $\mathcal{N}_{r,k}$ are finite, so $\mathcal{N}_r < \infty$ as well.

Let us recall orthogonality conditions (4.18) and (4.11) along with the definition of the Fourier transform (4.1). Clearly, we have

$$
G_{k,0,0} = 0, \quad G_{k,m,0,0} = 0, \quad 1 \leqslant k \leqslant N, \ \ m \in \mathbb{N}.
$$

Then due to bounds (4.2) and (4.4), we arrive at

$$
\left\| \frac{G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty} \leqslant 2\pi \|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)} \to 0, \quad m \to \infty,
$$

as we assumed, so (4.13) is valid. Note that (4.15) is an immediate consequence of (4.13) due to the standard triangle inequality.

Obviously, the estimate

$$\left| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right|$$
$$\leqslant \|G_{k,m,n_1,n_2} - G_{k,n_1,n_2}\|_{l^\infty} + \|n_2[G_{k,m,n_1,n_2} - G_{k,n_1,n_2}]\|_{l^\infty}$$

holds. Using formulas (4.2), (4.3) and (4.4), we derive the upper bound

$$\left\| \frac{(n_1^2 + n_2^2) G_{k,m,n_1,n_2}}{n_1^2 + |n_2|} - \frac{(n_1^2 + n_2^2) G_{k,n_1,n_2}}{n_1^2 + |n_2|} \right\|_{l^\infty}$$
$$\leqslant 2\pi \|G_{k,m}(x_1, x_2) - G_k(x_1, x_2)\|_{C(\Omega)}$$
$$+ \frac{1}{2\pi} \left\| \frac{\partial G_{k,m}(x_1, x_2)}{\partial x_2} - \frac{\partial G_k(x_1, x_2)}{\partial x_2} \right\|_{L^1(\Omega)} \to 0, \quad m \to \infty,$$

due to our assumptions. Hence, (4.14) is valid. Let us use the standard triangle inequality to establish that (4.16) follows easily from (4.14).

An easy limiting argument basing on (4.5), (4.6), (4.9), (4.10), (4.12), (4.15) and (4.16) yields (4.17). $\qquad\square$

Remark 4.3 The existence in the sense of sequences of solutions of system of equations (1.2) involving the transport terms will be considered in one of our future articles.

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Declarations

Conflict of interest The authors declare no competing interests.

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