Communications in Mathematics and Statistics
https://doi.org/10.1007/s40304-024-00414-5

Statistical inference for discretely observed fractional diffusion processes with random effects

Mohamed El Omari[1] · **Hamid El Maroufy**[5] · **Christiane Fuchs**[2,3,4]

Received: 28 March 2019 / Revised: 11 June 2019 / Accepted: 9 September 2024

Abstract
We address statistical inference for linear fractional diffusion processes with random effects in the drift. In particular, we investigate maximum likelihood estimators (MLEs) of the random effect parameters. First of all, we estimate the Hurst parameter $H \in (0, 1)$ from one single subject. Second, assuming that the Hurst index $H \in (0, 1)$ is known, we derive the MLEs and examine their asymptotic behavior as the number of subjects under study becomes large, with random effects being normally distributed.

Keywords Asymptotic normality · Fractional Brownian motion · Long-range memory process · Random effects model · Strong consistency

Mathematics Subject Classification 62F12 · 60G22

✉ Hamid El Maroufy
h.elmaroufy@usms.ma

Mohamed El Omari
mopzer.med@gmail.com

Christiane Fuchs
christiane.fuchs@uni-bielefeld.de

1 ESIM, Polydisciplinary Faculty of Sidi Bennour, Chouaib Doukkali University, El-Jadida 24000, Morocco

2 Faculty of Business Administration and Economics, Bielefeld University, 33615 Bielefeld, Germany

3 Helmholtz Zentrum München, Institute of Computational Biology, Neuherberg, Germany

4 Center for Mathematics, Technische Universität München, Garching, Germany

5 Mathematical Department, Faculty of Science and Technology, Sultan Moulay Slimane University, Béni-Mellal 23000, Morocco

Published online: 03 April 2025

1 Introduction

Parametric and nonparametric estimation of random effects models has recently been investigated by many authors (e.g., [1, 4–6, 19, 20]). In these models, noise is represented by Brownian motion characterized through the independence property of its increments. Such a property is not fulfilled for long-memory phenomena arising in a variety of applications from different scientific fields, including hydrology [17], biology [3], medicine [14], economics [9] or traffic networks [24]. As a consequence, self-similar processes have been used to successfully model data exhibiting long-range dependence. Among the simplest models that display long-range dependence, one can consider fractional Brownian motion (fBm), introduced to the statistics community by Mandelbrot and Van Ness [16]. A normalized fBm with Hurst index $H \in (0, 1)$ is a centered Gaussian process $\left(W_t^H \ : \ t \geq 0\right)$ with covariance

$$\mathbb{E}\left(W_s^H W_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Here, statistical estimation of model parameters is of particular importance and resulted in a growing number of papers devoted to statistical methods for equations with fractional noise. We will cite only a selection of them; further references can be found in [18, 22]. In [12], the authors proposed and studied maximum likelihood estimators (MLEs) for fractional Ornstein–Uhlenbeck processes. Related results were obtained by Prakasa [21], where a more general model was considered. In [10], the authors proposed a least squares (LS) estimator for fractional Ornstein–Uhlenbeck processes and proved its asymptotic normality. Recently, the same results were obtained for the fractional Vasicek model with long-memory using the same approach (LS) ([25]). Note that [11, 23] deal with the whole range of the Hurst parameter $H \in (0, 1)$. Meanwhile, we have cited other papers that consider only the case where $H > 1/2$, which corresponds to long-range dependence. Recall that for $H = 1/2$ we get a classical diffusion process extensively treated in the literature [15].

This paper deals with statistical estimation of population parameters for fractional stochastic differential equations (SDEs) with random effects. To our knowledge, this problem has not been investigated yet. Precisely, we consider fractional diffusion processes of the form

$$X_t = x + \int_0^t \left(a(X_s) + \phi b(X_s)\right) ds + W_t^H, \tag{1.1}$$

where ϕ is a random variable relying on a parameter θ to be estimated, and W^H is a normalized fBm with Hurst parameter H to be estimated. We study the additive linear case, $b(x) \equiv 1$, when $\phi \sim \mathcal{N}(\mu, \sigma^2)$. The estimators $\widehat{\mu}$, $\widehat{\sigma^2}$ and $\widehat{H}$ of μ, σ^2 and H, respectively, are constructed and their asymptotic behaviors are investigated. There are several reasons why we chose the model (1.1): It is simple, and we can derive explicit estimators. It generalizes the model considered in [11], while the techniques used here to investigate asymptotic properties are elementary due to the incorporation of the random effects, and hence, we avoid the Malliavin techniques. Second, (1.1) is widely applied in various fields. In fact, the Vasicek model is an example of type (1.1). The third reason is that the estimation of the population parameters requires few observations per subject, which coincides with several natural phenomena where the repeated measurements are rarely available if not impossible. Finally, nonparametric estimation has been realized recently by us for a similar model [8].

The paper is organized as follows. In Sect. 2, we introduce the model and some preliminaries about the likelihood function. In Sect. 3, we derive parameter estimators and establish consistency and asymptotic normality. Simulations are presented in Sect. 4, while Sect. 5 contains concluding remarks and gives directions of further research. Throughout the paper, the notations $\Longrightarrow$, $\overset{\mathbb{P}-as}{\Longrightarrow}$ and $\overset{\mathcal{D}}{\Longrightarrow}$ mean, respectively, simple convergence, convergence almost surely with respect to the probability measure $\mathbb{P}$ and convergence in distribution.

2 Model and Preliminary Results

Before introducing our estimation techniques, we first state some basic facts about fBm and its likelihood function. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^i), \mathbb{P}\right)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the $\mathbb{P}$-completion of the filtration generated by this process. Let $W^{H,i} = \left(W^{H,i}(t)\ ,\ t \leq T:\right)$, $i = 1, \cdots, N$ be N independent normalized fBm's with a common Hurst parameter $H \in (0, 1)$. Let $\phi_1, \cdots, \phi_N$ be N independent and identically distributed (i.i.d) $\mathbb{R}$-valued random variables on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ independent of $\left(W^{H,1}, \cdots, W^{H,N}\right)$. Consider N subjects $\left(X^i(t), \mathcal{F}_t^i, t \leq T\right)$ with dynamics ruled by the following general linear SDEs:

$$\begin{aligned} dX^i(t) &= \left(a(X^i(t)) + \phi_i b(X^i(t))\right) dt + dW^{H,i}(t) \\ X^i(0) &= x^i \in \mathbb{R}, \quad i = 1, \cdots, N, \end{aligned} \tag{2.1}$$

where $a(\cdot)$ and $b(\cdot)$ are supposed to be known in their own spaces. Let the random effects ϕ_i be $\mathcal{F}_0^i$-measurable with common density $g(\varphi, \theta) d\nu(\varphi)$, where ν is some dominating measure on $\mathbb{R}$ and θ is unknown parameter. Set $\theta \in U$, where U is an open set in $\mathbb{R}^d$. Sufficient conditions for the existence and uniqueness of solutions to (2.1) can be found in [18, p. 197] and references therein.

Let C_T denote the space of real continuous functions $(x(t)\ :\ t \in [0, T])$ defined on $[0, T]$ endowed with a σ-field $\mathcal{B}_T$. The σ-field $\mathcal{B}_T$ is associated with the topology of uniform convergence on $[0, T]$. We introduce the distribution $\mu_{X^i_{\varphi,H}}$ on $(C_T, \mathcal{B}_T)$ of the process $\left(X^i | \phi_i = \varphi\right)$. On $\mathbb{R} \times C_T$, $Q^i_{\theta,H} = g(\varphi, \theta) d\nu \otimes \mu_{X^i_{\varphi,H}}$ denotes the joint distribution of (ϕ_i, X^i). Let $\mathbb{P}^i_{\theta,H}$ be the marginal distribution of $\left(X^i(t)\ :\ t \leq T\right)$ on $(C_T, \mathcal{B}_T)$. Since the subjects are independent (this is inherited from the independence of ϕ_i and $W^{H,i}$), the distribution of the whole sample $\left(X^i(t)\ :\ t \leq T,\ i = 1, \cdots, N\right)$

on $C_T^{\otimes N}$ is defined by $\mathbb{P}_{\theta,H} = \otimes_{i=1}^N \mathbb{P}_{\theta,H}^i$. Thus, the likelihood can be defined as

$$\Lambda(\theta, H) = \frac{\mathrm{d}\mathbb{P}_{\theta,H}}{\mathrm{d}\mathbb{P}} = \prod_{i=1}^N \frac{\mathrm{d}\mathbb{P}_{\theta,H}^i}{\mathrm{d}\mathbb{P}^i},$$

where $\mathbb{P} = \otimes_{i=1}^N \mathbb{P}^i$ and $\mathbb{P}^i = \mu_{X^i_{\varphi_0,H}}$, provided that $\mu_{X^i_{\varphi,H}} \ll \mu_{X^i_{\varphi_0,H}}$ for some fixed $\varphi_0 \in \mathbb{R}$. It is well known that $\mu_{X^i_{\varphi,H}}$ coincides with the distribution of the process $X^{i,\varphi}$ defined by

$$dX^{i,\varphi}(t) = \Big(a(X^i(t)) + \varphi b(X^{i,\varphi}(t))\Big)\,dt + dW^{H,i}(t),\;\; X^{i,\varphi}(0) = x^i,$$

when $H = 1/2$, since in this case the process (X^i, ϕ_i) is Markovian (e.g., [7]); hence, the Girsanov formula can be applied to get the derivative $d\mu_{X^i_{\varphi,H}}/d\mu_{X^i_{\varphi_0,H}}$. For $H \neq 1/2$, the non-Markovian property of the coupled process (X^i, ϕ_i) makes the construction of the likelihood very difficult. In our case, however, the process X^i is transformed into a Y^i for which the law of $\Big(Y^i|\phi_i = \varphi\Big)$ coincides with the distribution of a parametric fractional diffusion process $Y^{i,\varphi}$.

3 Construction of Estimators and their Asymptotic Properties

Consider the following process

$$Y^i(t) := X^i(t) - x^i - \int_0^t a(X^i(s))ds, \quad t \geq 0 \tag{3.1}$$

$$= t\phi_i + W^{H,i}(t) \sim \mathcal{N}\left(t\mu, t^2\sigma^2 + t^{2H}\right), \quad t \geq 0. \tag{3.2}$$

Since ϕ_i and $W^{H,i}$ are independent, the process $\Big(Y^i(t) : t \in [0, T]\Big)$ is Gaussian. Furthermore, for each $\varphi \in \mathbb{R}$, we have $\mathbb{E}\left(Y^i(t)|\phi_i = \varphi\right) = t\varphi$ and $\mathbb{C}ov\left(Y^i(t), Y^i(s)|\phi_i = \varphi\right) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. For each subject Y^i, we consider n observations $Y^i := \left(Y^i(t_1), \cdots, Y^i(t_n)\right)'$ where $0 = t_0 < t_1 < \cdots < t_n = T$ is a subdivision of $[0, T]$. The density of Y^i given $\phi_i = \varphi$ is expressed as

$$\Pi(Y^i|\phi_i = \varphi, H) = \frac{1}{\sqrt{(2\pi)^n \det V(H)}} \exp\left(-\frac{1}{2}(Y^i - \varphi u)' V^{-1}(H)(Y^i - \varphi u)\right),$$

where $u = (t_1, \cdots, t_n)'$ and $(V(H))_{k,l} = \mathbb{C}ov\left(Y^i(t_k), Y^i(t_l)|\phi_i = \varphi\right)$ is the common covariance matrix of the subjects Y^i, $i = 1, \cdots, N$. The log-likelihood of the whole sample $\left(Y^1, \cdots, Y^N\right)$ is defined as

$$l(\theta, H) = \sum_{i=1}^{N} \log \int \Pi(Y^i|\phi_i = \varphi, H) g(\varphi, \theta) d\nu(\varphi). \tag{3.3}$$

For a specific distribution (say $g(\varphi, \theta)d\nu(\varphi) = \mathcal{N}(\mu, \sigma^2)$), we can solve the integrals given in (3.3). Indeed,

$$\int \Pi(Y^i|\phi_i = \varphi, H) g(\varphi, \theta) d\nu(\varphi) (2\pi)^{\frac{-n}{2}} \sigma^{-1} \det(V(H))^{\frac{-1}{2}} \left(u' V^{-1}(H) u + 1/\sigma^2\right)^{\frac{-1}{2}}$$
$$\times \exp\left[-\frac{1}{2}\left(\mu^2/\sigma^2 + Y^{i'} V^{-1}(H) Y^i - \frac{(u' V^{-1}(H) Y^i + \mu/\sigma^2)^2}{u' V^{-1}(H) u + 1/\sigma^2}\right)\right]. \tag{3.4}$$

3.1 Estimation of the Hurst Parameter *H*

Using data induced by one single subject (without loss of generality, say Y^1 with $t_j = j/n$, $j = 1, \cdots, n$, $T = 1$), we may construct a class of estimators of the Hurst index H. More precisely, for all $k > 0$ and for any filter $\gamma = (\gamma_0, \cdots \gamma_l)$ of order $p \geq 2$, that is,

$$\text{for all indices } 0 \leq r < p; \ \sum_{j=0}^{l} j^r \gamma_j = 0 \ \text{ and } \ \sum_{j=0}^{l} j^p \gamma_j \neq 0. \tag{3.5}$$

Consider the following arguments: $\widehat{H}(n, p, k, \gamma, Y^1) = g_{n,k,\gamma}^{-1}(S_n(k, \gamma))$, where $S_n(k, \gamma) = \frac{1}{n-l} \sum_{i=l}^{n-1} \left| \sum_{q=0}^{l} \gamma_q Y^1\left(\frac{i-q}{n}\right) \right|^k$, $g_{n,k,\gamma}(t) = \frac{1}{n^{tk}} \{\pi_t^\gamma(0)\}^{k/2} E_k$ and $\pi_t^\gamma(j) = -\frac{1}{2} \sum_{q,r}^{l} \gamma_q \gamma_r |q - r + j|^{2t}$, with $E_k = 2^{k/2} \Gamma(k + 1/2)/\Gamma(1/2)$, where $\Gamma(x)$ is the usual gamma function. For invertibility of the function $g_{n,k,\gamma}(\cdot)$, we refer to [2, p. 7].

Theorem 3.1 *The following statements hold true as the number of observations $n \longrightarrow \infty$:*

(i) $\widehat{H}(n, p, k, \gamma, Y^1) \overset{\mathbb{P}-as}{\Longrightarrow} H$,

(ii) $n^{-1/2} \log(n) \left(\widehat{H}(n, p, k, \gamma, Y^1) - H\right) \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}\left(0, \frac{A(H, k, \gamma)}{k^2}\right)$, *where*

$$A(t, k, \gamma) = \sum_{j \geq 1} (c_{2j}^k)^2 (2j)! \sum_{i \in \mathbb{Z}} \rho_t^\gamma(i)^{2j} \ \textit{with}$$

$$c_{2j}^{k} = \frac{1}{(2j)!}\prod_{q=0}^{j-1}(k-2q), \ \textit{and}\ \rho_t^{\gamma}(i) = \frac{\pi_t^{\gamma}(i)}{\pi_t^{\gamma}(0)}.$$

Proof Following Coeurjolly [2], we set $V^{\gamma}(i/n) = \sum_{q=0}^{l}\gamma_q W^{H,1}\left(\frac{i-q}{n}\right)$, for $i = l, \cdots, n-1$. From (3.5), we see that the filter γ is of order $p \geq 2$, $\sum_{q=0}^{l}\frac{i-q}{n}\gamma_q = 0$. Therefore, substituting $Y^1\left(\frac{i-q}{n}\right)$ by $\frac{i-q}{n}\phi_1 + W^{H,1}\left(\frac{i-q}{n}\right)$, we obtain

$$\begin{aligned} S_n(k,\gamma) &= \frac{1}{n-l}\sum_{i=l}^{n-1}\left|\sum_{q=0}^{l}\gamma_q Y^1\left(\frac{i-q}{n}\right)\right|^k \\ &= \frac{1}{n-l}\sum_{i=l}^{n-1}\left|\sum_{q=0}^{l}\gamma_q \frac{i-q}{n}\phi_1 + \sum_{q=0}^{l}\gamma_q W^{H,1}\left(\frac{i-q}{n}\right)\right|^k \\ &= \frac{1}{n-l}\sum_{i=l}^{n-1}\left|V^{\gamma}(i/n)\right|^k. \end{aligned}$$

Hence, our estimators coincide with estimators $\widehat{H}$ based on k-variations of the fBm (see [2, Proposition 2]) and the proof is complete. □

3.2 Estimation of the Population Parameter $\theta = (\mu, \sigma^2)$

Now, assume that H is known. From the log-likelihood given by (3.3) and (3.4), we derive an estimator $\widehat{\mu}$ given by

$$\widehat{\mu} = \frac{\frac{1}{N}\sum_{i=1}^{N} u'V^{-1}(H)Y^i}{u'V^{-1}(H)u}. \tag{3.6}$$

Derivation of an estimator for the parameter σ^2 is difficult. However, we can construct an alternative estimator and study its asymptotic behavior. Observing that $\widehat{\mu}$ is a sample mean drawn from a sequence of i.i.d random variables, one might think that sample variance could also be used to estimate σ^2. Unfortunately, simple computations show that such a sample variance is not consistent. Thus, as an alternative, we propose the following estimator for σ^2:

$$\widehat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{u'V^{-1}(H)Y^i}{u'V^{-1}(H)u}\right)^2 - \frac{1}{N^2}\left(\sum_{i=1}^{N}\frac{u'V^{-1}(H)Y^i}{u'V^{-1}(H)u}\right)^2 - \left(u'V^{-1}(H)u\right)^{-1}. \tag{3.7}$$

Theorem 3.2 *The estimator* $\widehat{\mu}$ *is unbiased,* $\widehat{\mu} \overset{\mathbb{P}-as}{\Longrightarrow} \mu$ *and* $\mathbb{V}ar(\widehat{\mu}) \longrightarrow 0$ *as* $N \to \infty$.

Proof Set $\epsilon^i = \left(W^{H,i}(t_1), \cdots, W^{H,i}(t_n)\right)'$. Substituting Y^i by $\phi_i u + \epsilon^i$, we have $\widehat{\mu} = \frac{1}{N}\sum_{i=1}^N \phi_i + \frac{\frac{1}{N}\sum_{i=1}^N u'V^{-1}(H)\epsilon^i}{u'V^{-1}(H)u}$, so $\mathbb{E}(\widehat{\mu}) = \mu$. For the second statement, we consider the random variables $\xi_i(H)$ defined by

$$\xi_i(H) = \frac{u'V^{-1}(H)Y^i}{u'V^{-1}(H)u}. \tag{3.8}$$

Clearly, $\xi_i(H)$ are i.i.d random variables with $\mathbb{E}(\xi_i(H)) = \mu < \infty$, then by the strong law of large numbers, $\widehat{\mu}$ converges almost surely to μ as $N \to \infty$. Set $z(H) := (z_1(H), \cdots, z_n(H)) = u'V^{-1}(H)$, we have

$$\begin{aligned}
\mathbb{V}ar(\widehat{\mu}) &= \mathbb{V}ar\left(\frac{1}{N}\sum_{i=1}^N \phi_i\right) + \frac{1}{N^2(z(H)\cdot u)^2}\mathbb{V}ar\left(\sum_{i=1}^N z(H)\cdot\epsilon^i\right) \\
&= \frac{1}{N^2}\mathbb{V}ar\left(\sum_{i=1}^N \phi_i\right) \\
&+ \frac{1}{N^2(z(H)\cdot u)^2}\sum_{i,j}^N \mathbb{E}\left\{\left(\sum_{k=1}^n z_k(H)W^{H,i}(t_k)\right)\left(\sum_{l=1}^n z_l(H)W^{H,j}(t_l)\right)\right\} \\
&= \frac{1}{N^2}\sum_{i=1}^N \mathbb{V}ar(\phi_i) + \frac{1}{N^2(z(H)\cdot u)^2}\sum_{i,j}^N\sum_{k,l}^n z_k(H)z_l(H)\mathbb{E}\left(W^{H,i}(t_k)W^{H,j}(t_l)\right) \\
&= \frac{\sigma^2}{N} + \frac{1}{N^2(z(H)\cdot u)^2}\sum_{i}^N\sum_{k,l}^n z_k(H)z_l(H)\mathbb{E}\left(W^{H,i}(t_k)W^{H,i}(t_l)\right) \\
&= \frac{\sigma^2}{N} + \frac{1}{N^2(z(H)\cdot u)^2}\sum_{i}^N\sum_{k,l}^n \frac{1}{2}z_k(H)z_l(H)\left(t_k^{2H} + t_l^{2H} - |t_k - t_l|^{2H}\right) \\
&= \frac{\sigma^2}{N} + \frac{1}{N^2(z(H)\cdot u)^2}Nz(H)V(H)z(H)' = \sigma^2 + \frac{Nu'V^{-1}(H)V(H)V^{-1}(H)u}{N^2(z(H)\cdot u)^2} \\
&= \frac{\sigma^2}{N} + \frac{1}{Nu'V^{-1}(H)u} \longrightarrow 0 \text{ as } N \to \infty.
\end{aligned}$$

□

Before considering the bias of $\widehat{\sigma^2}$, the estimator of σ^2, we first give the following result:

Lemma 3.3 *One has*

$$\mathbb{E}(\xi_1(H))^2 = \sigma^2 + \mu^2 + \frac{1}{u'V^{-1}(H)u} \text{ and } \mathbb{E}\left(\sum_{i=1}^N \xi_i(H)\right)^2 = N\sigma^2 + N^2\mu^2$$

$$+\frac{N}{u'V^{-1}(H)u},$$

where $\xi_i(H)$ *are random variables given by* (3.8).

Proof Substituting Y^1 by $\phi_1 u + \epsilon^1$, the independence of ϕ_1 and ϵ^1 gets

$$\begin{aligned}\mathbb{E}(\xi_1(H))^2 &= \mathbb{E}\left(\phi_1 + \frac{u'V^{-1}(H)\epsilon^1}{u'V^{-1}(H)u}\right)^2 \\ &= \mathbb{E}\phi_1^2 + \mathbb{E}\left(\frac{u'V^{-1}(H)\epsilon^1}{u'V^{-1}(H)u}\right)^2 \\ &= \sigma^2 + \mu^2 + \frac{1}{u'V^{-1}(H)u}.\end{aligned}$$

For the last equality, we used the same techniques as in the proof of Theorem 2. For the second statement, by using the previously defined random variables $z_i(H)'$s, we get

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{N}\xi_i(H)\right)^2 &= \mathbb{E}\left(\sum_{i=1}^{N}\phi_i + \sum_{i=1}^{N}\frac{z(H)\cdot\epsilon^i}{z(H)\cdot u}\right)^2 \\ &= \mathbb{E}\left(\sum_{i=1}^{N}\phi_i\right)^2 + \mathbb{E}\left(\sum_{i=1}^{N}\frac{z(H)\cdot\epsilon^i}{z(H)\cdot u}\right)^2 \\ &= \sum_{i=1}^{N}\mathbb{E}\phi_i^2 + 2\sum_{i<j}^{N}\mathbb{E}(\phi_i\phi_j) + \frac{1}{(u'V^{-1}(H)u)^2}\mathbb{V}ar\left(\sum_{i=1}^{N}z(H)\cdot\epsilon^i\right) \\ &= N\sigma^2 + N^2\mu^2 + \frac{N}{u'V^{-1}(H)u}.\end{aligned}$$

□

Theorem 3.4 *The estimator* $\widehat{\sigma^2}$ *is asymptotically unbiased,* $\widehat{\sigma^2} \overset{\mathbb{P}-as}{\Longrightarrow} \sigma^2$, *and* $\mathbb{V}ar(\widehat{\sigma^2}) = \frac{2(N-1)}{N^2}\left(\sigma^2 + \frac{1}{u'V^{-1}(H)u}\right)^2 \Longrightarrow 0$ *as* $N \to \infty$.

Proof By virtue of Lemma 3, we get

$$\begin{aligned}\mathbb{E}(\widehat{\sigma^2}) = \frac{1}{N}\sum_{i=1}^{N}\left(\sigma^2 + \mu^2 + \frac{1}{u'V^{-1}(H)u}\right) - \frac{1}{N^2}\left(N\sigma^2 + N^2\mu^2 + \frac{N}{u'V^{-1}(H)u}\right) \\ - \frac{1}{u'V^{-1}(H)u} \\ = \frac{N-1}{N}\sigma^2 - \frac{1}{N(u'V^{-1}(H)u)} \Longrightarrow \sigma^2 \text{ as } N \to \infty.\end{aligned}$$

Applying the strong law of large numbers and the continuous mapping theorem for almost sure convergence, we get

$$\begin{aligned}
\widehat{\sigma^2} &= \frac{1}{N}\sum_{i=1}^{N}\xi_i(H)^2 - \left(\frac{1}{N}\sum_{i=1}^{N}\xi_i(H)\right)^2 - \frac{1}{u'V^{-1}(H)u} \\
&\overset{\mathbb{P}-as}{\Longrightarrow} \mathbb{E}(\xi_1(H))^2 - \mathbb{E}^2(\xi_1(H)) - \frac{1}{u'V^{-1}(H)u} = \mathbb{V}ar(\xi_1(H)) - \frac{1}{u'V^{-1}(H)u} \\
&= \mathbb{V}ar\left(\phi_1 + \frac{u'V^{-1}(H)\epsilon^1}{u'V^{-1}(H)u}\right) - \frac{1}{u'V^{-1}(H)u} \\
&= \mathbb{V}ar\phi_1 + \mathbb{V}ar\left(\frac{u'V^{-1}(H)\epsilon^1}{u'V^{-1}(H)u}\right) - \frac{1}{u'V^{-1}(H)u} \\
&= \sigma^2 + \mathbb{E}\left(\frac{u'V^{-1}(H)\epsilon^1}{u'V^{-1}(H)u}\right)^2 - \frac{1}{u'V^{-1}(H)u} = \sigma^2.
\end{aligned}$$

Similar computations lead to

$$\begin{aligned}
\mathbb{V}ar(\widehat{\sigma^2}) &= \frac{N-1}{N^3}\left((N-1)\mathbb{E}(\xi_1(H)-\mu)^4 - (N-3)\beta^2\right) \\
&= \frac{2(N-1)}{N^2}\beta^2,
\end{aligned}$$

where $\beta = \mathbb{V}ar(\xi_1(H)) = \sigma^2 + \dfrac{1}{u'V^{-1}(H)u}$. In the last equality we used the fact that $(\xi_1(H) - \mu)$ is a centered Gaussian random variables with variance β. □

Remark For the case of continuous observation with horizon T, we propose the following estimator $\widetilde{\mu}(T, N)$ defined by

$$\widetilde{\mu}(T, N) = \frac{1}{NT}\sum_{i=1}^{N} Y^i(T).$$

It is easy to see that $\mathbb{E}\left|\frac{1}{T}Y^i(T) - \phi_i\right|^2 \le \frac{1}{T^{2-2H}} \longrightarrow 0$ as $T \longrightarrow \infty$ and $\widetilde{\mu}(T, N)$ is consistent when $T, N \to \infty$. The reason for choosing this double asymptotic framework is that we proceed in two steps: In the first step, we estimate random effects ϕ_i as the horizon T increases to ∞; then, we use the empirical mean and variance to estimate $\theta = (\mu, \sigma^2)$, where the random effects are replaced by their estimators.

Theorem 3.5 *The estimators $\widehat{\mu}$ and $\widehat{\sigma^2}$ are asymptotically normal, i.e.,*

$$\sqrt{N}\,(\widehat{\mu} - \mu) \overset{\mathcal{D}}{\Longrightarrow} \mathcal{N}\left(0, \sigma^2 + \frac{1}{u'V^{-1}(H)u}\right) \text{ as } N \to \infty, \tag{3.9}$$

and

$$\sqrt{\frac{N}{2}}\left(\widehat{\sigma^2}-\sigma^2\right)\stackrel{\mathcal{D}}{\Longrightarrow}\mathcal{N}\left(0,\left(\sigma^2+\frac{1}{u'V^{-1}(H)u}\right)^2\right)\ as\ N\to\infty. \tag{3.10}$$

Proof Since $\widehat{\mu}$ is the average of N i.i.d random variables with finite mean and finite variance, (3.9) follows immediately from the central limit theorem. In order to show (3.10), we consider the following random variables $\widetilde{\xi}_i(H)=\sqrt{\frac{N}{N-1}}(\xi_i(H)-\widehat{\mu})$, $i=1,\cdots,N$ and set $\beta=\sigma^2+\frac{1}{u'V^{-1}(H)u}$. We see that $\left(\widetilde{\xi}_i(H)\ ,\ i=1,2,\cdots\right)$ is a centered Gaussian process with variance $\mathbb{V}ar(\widetilde{\xi}_i(H))=\mathbb{E}(\widetilde{\xi}_i(H)^2)=\beta$, and $\mathbb{V}ar(\widetilde{\xi}_i(H)^2)=2\beta^2$. So using the strong law of large numbers, we have $\widetilde{\sigma^2}=\frac{1}{N}\sum_{i=1}^{N}\widetilde{\xi}_i(H)^2\stackrel{\mathbb{P}-as}{\Longrightarrow}\beta$ as $N\to\infty$. Furthermore, the central limit theorem leads to $\sqrt{N}\left(\widetilde{\sigma^2}-\beta\right)\stackrel{\mathcal{D}}{\Longrightarrow}\mathcal{N}(0,2\beta^2)$ as $N\to\infty$. Since $\sqrt{\frac{N}{2}}\left(\widehat{\sigma^2}-\sigma^2\right)=\alpha_N\sqrt{N}\left(\widetilde{\sigma^2}-\beta\right)-\varepsilon_N$, where $\alpha_N=\frac{N-1}{\sqrt{2}N}$ and $\varepsilon_N=\frac{\beta}{\sqrt{2}N}$, therefore, using Slutsky theorem, the convergence in (3.10) is easily concluded. □

4 Simulations

We implemented the two studied population parameter estimators to investigate their empirical behavior. To that end, we simulated the observed vectors Y^i using (3.2) for two numbers of subjects $N=50$ and $N=500$ with different lengths of observations per subject; $n=2^2$, $n=2^5$ and $n=2^8$. The fBm's were simulated as in [13]. The experimental design looks as follows: We set H equal to 0.15, 0.5 and 0.85. For each case, replications involving 400 samples are obtained by resampling n trajectories of Y^i.

Table 1 The means with exact (bold) and empirical (italic) standard deviations of estimators $\widehat{\mu}, \widehat{\sigma^2}$ based on 400 samples, with true values $(\mu_0,\sigma_0^2)=(-2,1)$, $\boldsymbol{(T,n)=(5,2^2)}$, and different values of N (= 50; 500)

True values	$H=0.15$	$H=0.50$	$H=0.85$
$N=50$	Mean (std. dev.)	Mean (std. dev.)	Mean and std. dev.'s
$\mu=-2$	−1.9902(**0.1456** *0.1430*)	−1.9964(**0.1549** *0.1594*)	− 1.9820(**0.1795** *0.2009*)
$\sigma^2=1$	0.9744(**0.2099** *0.1942*)	1.0303(**0.2376** *0.2494*)	1.3314(**0.3191** *0.3891*)
$N=500$			
$\mu=-2$	−2.0009(**0.0460** *0.0441*)	−1.9986(**0.0490** *0.0515*)	−1.9985(**0.0568** *0.0634*)
$\sigma^2=1$	0.9964(**0.0670** *0.0689*)	1.0442(**0.0758** *0.0836*)	1.2022(**0.1018** *0.1228*)

(For interpretation of the references to color in this table the reader is referred to the electronic version of this article)

Table 2 The means with exact (bold) and empirical (italic) standard deviations of estimators $\widehat{\mu}, \widehat{\sigma^2}$ based on 400 samples, with true values $(\mu_0, \sigma_0^2) = (-2, 1)$, $\boldsymbol{(T, n) = (5, 2^5)}$, and different values of N (= 50; 500)

True values $N = 50$	$H = 0.15$ Mean (std. dev.)	$H = 0.50$ Mean (std. dev.)	$H = 0.85$ Mean and std. dev.'s
$\mu = -2$	−2.0050(**0.1449** *0.1427*)	−2.0146(**0.1549** *0.1518*)	− 1.9824(**0.1793** *0.1920*)
$\sigma^2 = 1$	0.9713(**0.2077** *0.2075*)	1.0028(**0.2376** *0.2247*)	1.0871(**0.3181** *0.3391*)
$N = 500$			
$\mu = -2$	−2.0057(**0.0458** *0.0434*)	−1.9979(**0.0490** *0.0498*)	−2.0038(**0.0567** *0.0596*)
$\sigma^2 = 1$	1.0005(**0.0663** *0.0671*)	1.0021(**0.0758** *0.0758*)	1.0849(**0.1015** *0.1011*)

(For interpretation of the references to color in this table the reader is referred to the electronic version of this article)

Table 3 The means with exact (bold) and empirical (italic) standard deviations of estimators $\widehat{\mu}, \widehat{\sigma^2}$ based on 400 samples, with true values $(\mu_0, \sigma_0^2) = (-2, 1)$, $\boldsymbol{(T, n) = (5, 2^8)}$, and different values of N (= 50; 500)

True values $N = 50$	$H = 0.15$ Mean (std. dev.)	$H = 0.50$ Mean (std. dev.)	$H = 0.85$ Mean and std. dev.'s
$\mu = -2$	−2.0015(**0.1447** *0.1454*)	− 1.9960(**0.1549** *0.1563*)	− 2.0055(**0.1792** *0.1709*)
$\sigma^2 = 1$	0.9996(**0.2073** *0.2008*)	0.9764(**0.2376** *0.2448*)	0.9971(**0.3180** *0.3323*)
$N = 500$			
$\mu = -2$	−1.9997(**0.0458** *0.0442*)	−2.0009(**0.0490** *0.0471*)	− 2.0006(**0.0567** *0.0566*)
$\sigma^2 = 1$	0.9971(**0.0662** *0.0650*)	0.9993(**0.0758** *0.0747*)	1.0083(**0.1015** *0.1045*)

(For interpretation of the references to color in this table, the reader is referred to the electronic version of this article)

The averages of the estimators and their exact against empirical standard deviations are reported in Tables 1–3. The tables show that the parameter estimates are generally much closer to their true values as the number of subjects increases. Figures 1–3 display histograms of the estimates, which reveal empirical convergence toward a limit distribution as N is sufficiently large. This confirms what was established before. From Table 1, we see that the estimates of σ^2 deviate from underlying true values when there are very few observations ($n \leq 2^3$) per subject when $H = 0.85$. This situation appears whenever H becomes larger than $1/2$. In this situation, for non-synthetic cases where the true value of σ^2 is not available, it will be better to choose n as large as possible ($n \geq 2^4$), but this leads to huge computational cost for large values of N, since, to keep the balance between the computational cost and goodness of fit, a small value of n and sufficiently large values of N should be considered.

5 Concluding Remarks

In this paper, we have provided full parametric likelihood estimation of population parameters for a specific dynamical model described by a fractional SDE including random effects in the drift. We are essentially concerned with the estimation of the

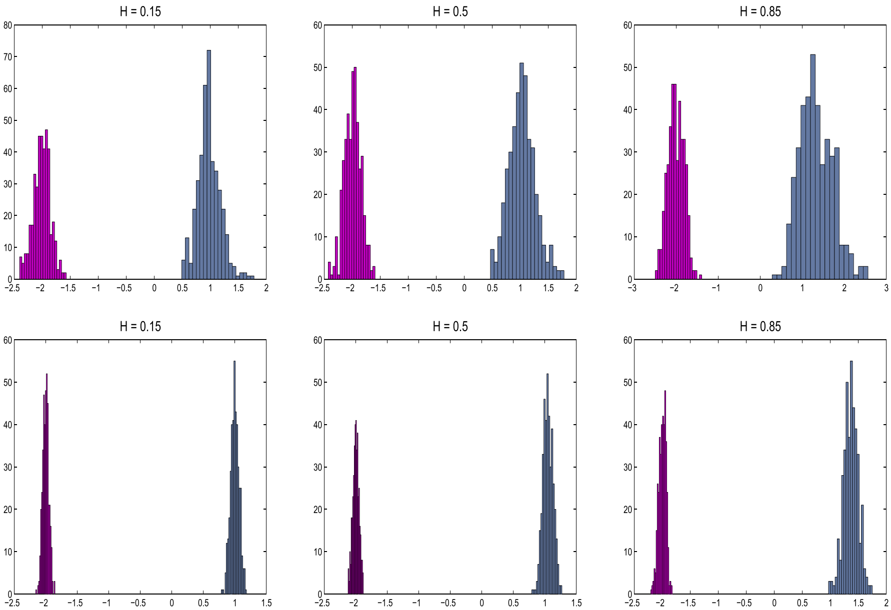

Fig. 1 Frequency histograms of population parameter estimates based on 400 samples for different values of (N, H). In each box of the two rows (top $N = 50$ and bottom $N = 500$), histograms of $\widehat{\mu}$ (pink) and $\widehat{\sigma^2}$ (gray) are given for fixed parameters $(\mu, \sigma^2, T, \boldsymbol{n}) = (-2, 1, 5, \mathbf{2^2})$. (For interpretation of the references to color in the legend of this figure, the reader is referred to the electronic version of this article)

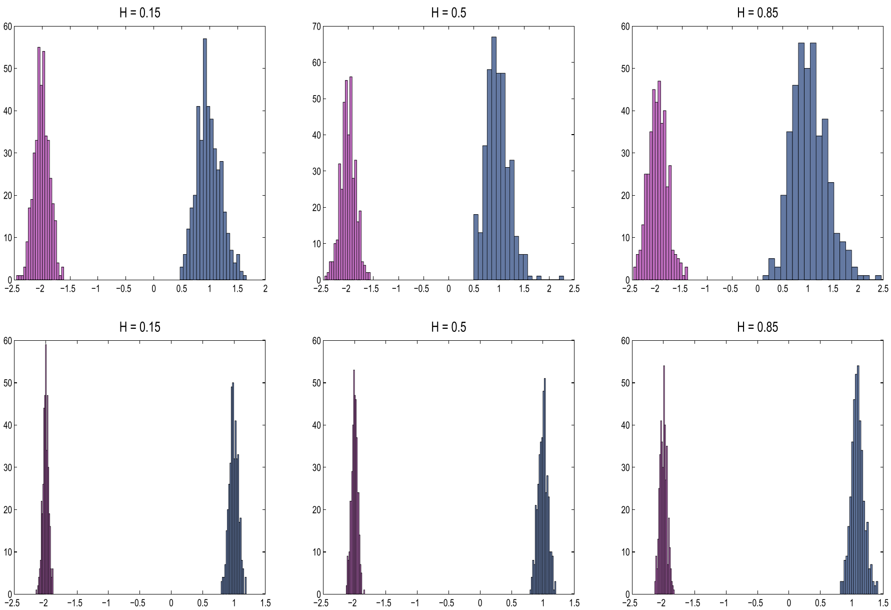

Fig. 2 Frequency histograms of population parameter estimates based on 400 samples for different values of (N, H). In each box of the two rows (top $N = 50$ and bottom $N = 500$), histograms of $\widehat{\mu}$ (pink) and $\widehat{\sigma^2}$ (gray) are given for fixed parameters $(\mu, \sigma^2, T, \boldsymbol{n}) = (-2, 1, 5, \mathbf{2^5})$. (For interpretation of the references to color in the legend of this figure, the reader is referred to the electronic version of this article)

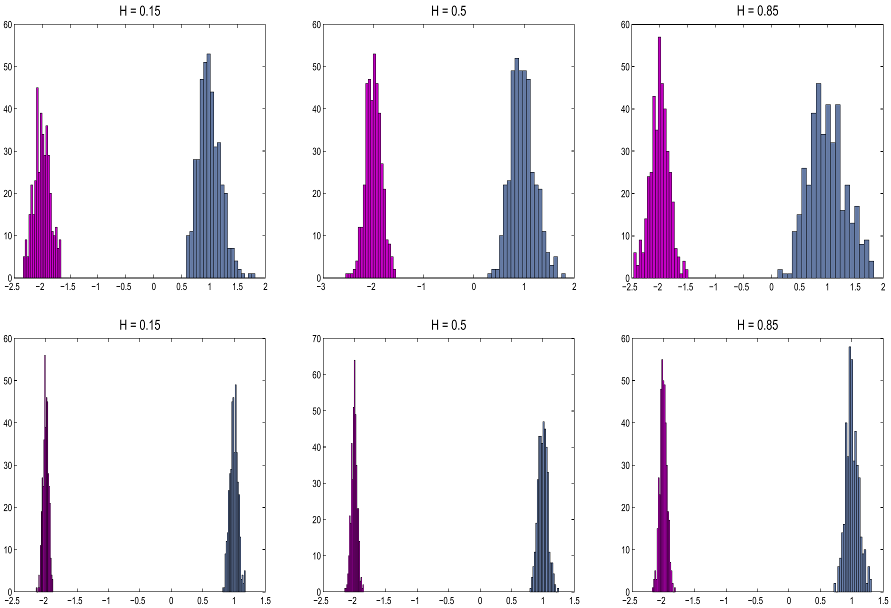

Fig. 3 Frequency histograms of population parameter estimates based on 400 samples for different values of (N, H). In each box of the two rows (top $N = 50$ and bottom $N = 500$), histograms of $\widehat{\mu}$ (pink) and $\widehat{\sigma^2}$ (gray) are given for fixed parameters $(\mu, \sigma^2, T, \boldsymbol{n}) = (-2, 1, 5, \mathbf{2^8})$. (For interpretation of the references to color in the legend of this figure, the reader is referred to the electronic version of this article)

Hurst index, as well as with the mean and variance estimators of the random effects that have a Gaussian distribution. Qualitative and asymptotic properties of the estimators are obtained when the population of subjects becomes large.

This study suggests several important directions for future research. First, asymptotic properties of the maximum likelihood estimators for μ and σ^2 remain open when the Hurst index H is unknown. Given that the model is fully parameterized, one may wish to estimate H, μ and σ^2 simultaneously. The achievement of this task is part of our ongoing work. Second, the present study assumes that the model is linear and the diffusion is constant and equal to one. This assumption is not verified in almost all real applications. So to overcome this issue, one can use, for example, Euler scheme approximations. However, it is not clear how to get an explicit approximation for the maximum likelihood function. Such an extension would be worth being studied from both theoretical and applied points of view. Third, as mentioned previously, we may estimate the population parameters by using a double asymptotic framework. Such an idea is considered in an ongoing work for a more general model and in the nonparametric estimation context.

Acknowledgements The anonymous referees and the editors are acknowledged for their constructive comments and suggestions, which have greatly improved the paper. The three authors acknowledge the financial support of the Moroccan-German Scientific and Technics Programme; Project: **PMARS III** 060/2015.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

References

1. Antic, J., Laffont, C.M., Chafai, D., Condorcet, D.: Comparison of nonparametric methods in nonlinear mixed effects models. Comput. Statist. Data Anal. **53**, 642–656 (2009)
2. Coeurjolly, J.F.: Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. Stat. Inference Stoch. Process **4**, 199–227 (2001)
3. Collins, J.J., De Luca, C.J.: Upright, correlated random walks: a statistical-biomechanics approachto the human postural control system. Chaos **5**(1), 57–63 (1995)
4. Comte, F., Samson, A.: Nonparametric estimation of random effects densities in linear mixed-effects model. J. Nonparametr. Stat **24**(4), 951–975 (2012)
5. Delattre, M., Genon-Catalot, V., Samson, A.: Maximum likelihood estimation for stochastic differential equations with random effects. Scand. J. Stat. **40**, 322–343 (2012)
6. Ditlevsen, S., De Gaetano, A.: Mixed effects in stochastic differential equation models. REVSTAT - Statist. J. **3**, 137–153 (2005)
7. Genon-Catalot, V., Laredo, C.: Stationary distributions for stochastic differential equations with random effects and statistical applications. MAP5 -09(2013)
8. El Omari, M., El Maroufy, H. Fuchs, C.: Non-parametric estimation for fractional diffusion processes with random effects, Statistics, (2019)
9. Granger, C.W.J.: The typical spectral shape of an economic variable. Econometrica **34**, 150–161 (1966)
10. Hu, Y., Nualart, D.: Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statist. Probab. Lett **80**(11–12), 1030–1038 (2010)
11. Hu, Y., Nualart, D., Xiao, W., Zhang, W.: Exact maximum likelihood estimator for drift fractional Brownian motion at discrete observation. Acta Math. Sci. Ser. B Engl. Ed **31**(5), 1851–1859 (2011)
12. Kleptsyna, M.L., Le Breton, A.: Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Stat. Inference Stoch. Process **5**(3), 229–248 (2002)
13. Kroese, D. P., Botev, Z.I.: Spatial process generation.(2013). available arXiv:1308.0399v1
14. Kuklinski, W.S., Chandra, K., Ruttimann, U.E.: Application of fractal texture analysis to segmentation of dental radiographs SPIE. Med. Imaging III: Image Proc. **1092**, 111–117 (1989)
15. Liptser, R., Shiryaev, A.: Statistics of random processes II: general theory. Springer, New York (2001)
16. Mandelbort, B., Van Ness, W.: Fractional Brownian motions, fractional noises and applications. SIAM Review **10**, 422–437 (1968)
17. Mcleod, A.I., Hipel, K.W.: Preservation of the rescaled adjusted range: a reassessment of the hurst exponent water resources. Research **14**(3), 491–508 (1978)
18. Mishura, Y.S.: Stochastic calculus for fractional Brownian motion and related processes, Lecture Notes in Mathematics, vol. 1929. Springer, Berlin (2008)
19. Nie, L.: Convergence rate of the MLE in generalized linear and nonlinear mixed-effects models: theory and applications. J. Statist. Plann. Inference **137**, 1787–1804 (2007)
20. Picchini, U., Ditlevsen, S.: Practicle estimation of a high dimensional stochastic differential mixed-effects models. Comput. Statist. Data. Anal **55**, 1426–1444 (2011)
21. Prakasa Rao, B.L.S.: Parametric estimation for linear stochastic differential equations driven by fractional Brownian motion. Random Oper. Stoch. Equ **11**(3), 229–242 (2003)
22. Prakasa Rao, B.L.S.: Statistical Inference for Fractional Diffusion Processes. Wiley Series in Probability and Statistics, Wiley, Chichester (2010)
23. Tudor, C.A., Viens, F.G.: Statistical aspects of the fractional stochastic calculus. Ann. Stat. **35**(3), 1183–1212 (2007)
24. Willinger, W., Taqqu, M.S., Leland, W.E., Wilson, D.V.: Self-similarity in high speed packet traffic: analysis and modelisation of ethernet traffic measurements. Stat. Sci. **10**, 67–85 (1995)
25. Xiao, W., Yu, J.: Asymptotic theory for estimating drift parameters in the fractional Vasicek model. Econ. Theory **35**, 1–34 (2018)