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Regularized Shannon Sampling Formulas Related to the Special Affine Fourier Transform

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Abstract

In this paper, we present new regularized Shannon sampling formulas related to the special affine Fourier transform (SAFT). These sampling formulas use localized sampling with special compactly supported, continuous window functions, namely B-spline, sinh-type, and continuous Kaiser–Bessel window functions. In contrast to the known Shannon sampling series for SAFT, the regularized Shannon sampling formulas for SAFT possesses an exponential decay of the approximation error and are numerically robust in the presence of noise, if certain oversampling condition is fulfilled.

Keywords Special affine Fourier transform · SAFT · Shannon sampling theorem · Compactly supported window function · Regularized Shannon sampling formula related to SAFT · Error estimates · Numerical robustness

Mathematics Subject Classification 94A20 · 42A38 · 65T50

1 Introduction

The *special affine Fourier transform* (SAFT) was introduced by Abe and Sheridan [1] for the study of certain operations on optical wave functions. For $f \in L^1(\mathbb{R})$, it is an

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integral transform of the form

$$\mathscr{F}_A f(\omega) = \int_{\mathbb{R}} f(t)\, \phi_A(t,\omega)\, \mathrm{d}t, \quad \omega \in \mathbb{R}, \tag{1.1}$$

with the kernel

$$\phi_A(t,\omega) = \frac{1}{\sqrt{2\pi\,|b|}} \exp\left[\frac{\mathrm{i}}{2b}\left(at^2 + 2pt - 2\omega t + d\omega^2 + 2(bq - dp)\omega\right)\right], \quad t,\, \omega \in \mathbb{R},$$

depending on a parameter matrix $A = \begin{pmatrix} a & b & p \\ c & d & q \end{pmatrix} \in \mathbb{R}^{2\times 3}$ with submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $ad - cb = 1$ and $b \neq 0$, and an offset vector $\begin{pmatrix} p \\ q \end{pmatrix}$. The name "special affine Fourier transform" comes from the fact that the transform (1.1) is related to a special affine transform of the time-frequency coordinates

$$\begin{pmatrix} t' \\ \omega' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ \omega \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belonging to the special linear group $SL(2,\mathbb{R})$. We will not go into a detailed discussion of the origin and relations of the SAFT to various fields in physics, but we take (1.1) merely as a signal transform. The SAFT is a generalization of the classical classical *Fourier transform* $\mathscr{F}$ defined by

$$\mathscr{F} f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{1.2}$$

and it includes a number of well-known signal transforms as special cases (see Table 1 or [8, Table 1]).

As a matter of fact the ordinary translation operator $T_x f(t) = f(t-x)$ does not interact nicely with the kernel of the SAFT (1.1), i.e., the function $T_x\phi_A(t,\omega) = \phi_A(t-x,\omega)$ is in general different from $\phi_A(t,\omega)\,\phi_A(-x,\omega)$ except if $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. As a consequence, working with the ordinary translation operator a number of facts known from ordinary Fourier analysis are no longer valid. This holds in particular for the important Shannon sampling theorem, which allows the reconstruction of a function $f \in L^2(\mathbb{R})$ with Fourier transform $\mathscr{F} f$ supported in $[-\pi L, \pi L]$ with some $L > 0$ from its samples $f\left(\frac{n}{L}\right)$, $n \in \mathbb{Z}$, using ordinary translates of the scaled cardinal sine function $\operatorname{sinc}(L\,\cdot)$, viz.

Table 1 Various parameter matrices and corresponding SAFT's

Parameter matrix A	Condition	Corresponding SAFT $\mathscr{F}_A$
$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$		Fourier transform
$\begin{pmatrix} 0 & 1 & p \\ -1 & 0 & q \end{pmatrix}$		Offset Fourier transform
$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix}$	$ad - cb = 1,\ abd \neq 0$	Canonical linear transform
$\begin{pmatrix} a & b & p \\ c & d & q \end{pmatrix}$	$ad - cb = 1,\ abd \neq 0$	Offset canonical linear transform
$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$b \neq 0$	Fresnel transform
$\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \end{pmatrix}$	$\theta \notin \pi\mathbb{Z}$	Fractional Fourier transform
$\begin{pmatrix} \cos(\theta) & \sin(\theta) & p \\ -\sin(\theta) & \cos(\theta) & q \end{pmatrix}$	$\theta \notin \pi\mathbb{Z}$	Offset fractional Fourier transform
$\begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 \end{pmatrix}$	$\theta \neq 0$	Hyperbolic transform

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{L}\right) \operatorname{sinc}(Lt - n), \quad t \in \mathbb{R}. \tag{1.3}$$

In case that the SAFT $\mathscr{F}_A$ of $f \in L^2(\mathbb{R})$ is supported in the interval $[-\pi L\,|b|, \pi L\,|b|]$, formula (1.3) does in general no longer hold. In order to get an analogue for (1.3) in case of the SAFT it is necessary to work with a different concept of translations. In [10] a generalized A-translation operator T_x^A was introduced which matches with the SAFT $\mathscr{F}_A$, and its various consequences for the related harmonic analysis were studied. This operator reads as

$$T_x^A f(t) = \mathrm{e}^{-\mathrm{i}\frac{a}{b}x(t-x)} f(t-x), \quad t,\ x \in \mathbb{R}.$$

It obviously reduces to the ordinary translation if $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Moreover, it was demonstrated in [10] that fundamental properties of the ordinary Fourier transform, as for example the convolution theorem, hold for the SAFT as well, if we work with the operator T_x^A instead of T_x. In [7] Bhandari and Zayed also studied the SAFT in some detail and worked out a number of aspects of this transform. The authors considered a generalized translation somewhat implicitly but did not work out its properties and consequences with respect to the related harmonic analysis explicitly. In [9] the SAFT and related modulation spaces were extensively studied. Furthermore, in [7] and later in [10] shift invariant spaces related to T_x^A were studied and, in particular, an analogue of the Shannon sampling theorem for the SAFT $\mathscr{F}_A$ was derived. It states that a function $f \in L^2(\mathbb{R})$, where the *support of* $\mathscr{F}_A f$

$$\operatorname{supp}(\mathscr{F}_A f) = \operatorname{clos}\{\omega \in \mathbb{R} : \mathscr{F}_A f(\omega) \neq 0\}$$

is contained in $[-\pi L\,|b|, \pi L\,|b|]$ with some $L > 0$, can be reconstructed from its samples $f\left(\frac{n}{T}\right)$, $n \in \mathbb{Z}$, with $T \geq L$ as

$$f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt - n), \quad t \in \mathbb{R}, \tag{1.4}$$

with the exponential function

$$\rho_A(t) = \mathrm{e}^{\frac{\mathrm{i}}{2b}\,(at^2+2pt)}, \quad t \in \mathbb{R}.$$

Although this representation is quite satisfactory from a theoretical point of view it suffers in the same manner as the ordinary Shannon sampling formula (1.3) from some computational shortcomings. Apart from the obvious problem of using infinitely many samples, the slow decay of the scaled cardinal sine function $\operatorname{sinc}(T\,\cdot)$ prevents a good approximation of f by truncation of the series. In order to mitigate these flaws, the scaled cardinal sine function $\operatorname{sinc}(T\,\cdot)$ is multiplied by a suitable compactly supported, continuous window function φ. Hence instead of $\operatorname{sinc}(T\,\cdot)$, the function $\operatorname{sinc}(T\,\cdot)\,\varphi$ is used for the reconstruction of a function $f \in L^2(\mathbb{R})$ with $\operatorname{supp}(\mathscr{F}_A f) \subseteq [-\pi L\,|b|, \pi L\,|b|]$. In the classical situation $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, localization by a suitable window function and oversampling with $T > L$ was used as regularization strategies for the problem. There is a huge stack of papers dealing with the problem of a stable and robust computation of the sampling series, see [11, 14, 15, 20, 26, 27] and references therein. The Gaussian window function was frequently used as a window function. However, it turned out that certain compactly supported, continuous window function lead to much better approximation results [14]. Motivated by this observation, we concentrate our studies in this paper on a suitable class of compactly supported, continuous window functions as well. We consider in particular three special window functions in detail and analyse the corresponding approximation behaviour.

The paper is organized as follows. In Sect. 2 we recall all necessary properties of the SAFT. Section 3 provides the results on sampling for the SAFT $\mathscr{F}_A$. Regularization of the Shannon sampling formula for $\mathscr{F}_A$ will be considered in Sect. 4. A detailed study of regularization by specific window functions will be presented in Sect. 5.

2 Special Affine Fourier Transform

In this section we shall define the SAFT more precisely and we will discuss the relevant properties of this transform. Before doing so, let us briefly recall the definition of function spaces which are relevant for us in the sequel. By $C_0(\mathbb{R})$ we denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ which vanish at infinity equipped with the norm $\|f\|_\infty = \max_{t\in\mathbb{R}} |f(t)|$. The spaces $L^p(\mathbb{R}),\ 1 \leq p \leq \infty$ are defined as usual with their respective norms

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{R}} |f(t)|^p \, \mathrm{d}t\right)^{1/p} & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t)| & p = \infty. \end{cases}$$

The inner product for $L^2(\mathbb{R})$ is denoted by $\langle \cdot, \cdot \rangle$.

The special affine Fourier transform (SAFT) of a function $f \in L^1(\mathbb{R})$ is defined as

$$\mathscr{F}_A f(\omega) = \frac{1}{\sqrt{2\pi\,|b|}} \int_{\mathbb{R}} f(t)\, \mathrm{e}^{\frac{\mathrm{i}}{2b}\left(at^2+2pt-2\omega t+d\omega^2+2(bq-dp)\omega\right)} \,\mathrm{d}t, \quad \omega \in \mathbb{R}, \tag{2.1}$$

where $A = \begin{pmatrix} a & b & p \\ c & d & q \end{pmatrix} \in \mathbb{R}^{2\times 3}$ is a fixed parameter matrix with $ad - bc = 1$ and $b \neq 0$. We exclude the case $b = 0$, which is discussed in [8] too. For comprehensive studies of the properties of the SAFT we refer to [7–10].

The SAFT $\mathscr{F}_A$ can be written in a more convenient form using the auxiliary functions

$$\eta_A(\omega) = \mathrm{e}^{\frac{\mathrm{i}}{2b}(d\omega^2+2(bq-dp)\omega)}, \quad \rho_A(t) = \mathrm{e}^{\frac{\mathrm{i}}{2b}(at^2+2pt)}, \quad \omega,\ t \in \mathbb{R}. \tag{2.2}$$

The SAFT $\mathscr{F}_A$ now reads as

$$\mathscr{F}_A f(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}}\, \mathscr{F}(\rho_A f)\left(\frac{\omega}{b}\right), \quad \omega \in \mathbb{R}, \tag{2.3}$$

where $\mathscr{F}$ stands for the ordinary Fourier transform (1.2). As η_A and ρ_A are unimodular functions, i.e., $|\eta_A(\omega)| = 1 = |\rho_A(t)|$ for all $t,\ \omega \in \mathbb{R}$, we immediately get from (2.3) that $\mathscr{F}_A f$ belongs to $C_0(\mathbb{R})$ with $\|\mathscr{F}_A f\|_\infty \le (2\pi|b|)^{-1/2}\,\|f\|_1$. Moreover, (2.3) also shows that $\mathscr{F}_A$ can be extended to $L^2(\mathbb{R})$ and defines a unitary operator on that space, viz.

$$\langle \mathscr{F}_A f, \mathscr{F}_A g\rangle = \langle f, g\rangle, \quad f,\ g \in L^2(\mathbb{R}). \tag{2.4}$$

In particular, $\|\mathscr{F}_A f\|_2 = \|f\|_2$ for $f \in L^2(\mathbb{R})$. The inverse of $\mathscr{F}_A$ on $L^2(\mathbb{R})$ can readily be obtained from (2.3). It is given as

$$\mathscr{F}_A^{-1} f(t) = \frac{\overline{\rho_A(t)}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(\omega)\, \overline{\eta_A(\omega)}\, \mathrm{e}^{\mathrm{i}\omega t/b} \,\mathrm{d}\omega. \tag{2.5}$$

The inverse of the SAFT $\mathscr{F}_A$ can be represented in the form

$$\mathscr{F}_A^{-1} f(t) = \int_{\mathbb{R}} f(\omega)\, \phi_{A'}(\omega, t) \,\mathrm{d}\omega = \mathscr{F}_{A'} f(t), \quad t \in \mathbb{R},$$

with kernel

$$\begin{aligned}\phi_{A'}(\omega, t) &= \frac{1}{\sqrt{2\pi\,|b|}} \exp\Big[-\frac{\mathrm{i}}{2b}\left(d\omega^2 + 2\,(pq - dp) - 2\omega t + at^2 + 2pt\right)\Big] \\ &= \frac{1}{\sqrt{2\pi\,|b|}}\,\overline{\eta_A(\omega)}\,\overline{\rho_A(t)}\,\mathrm{e}^{\mathrm{i}\,\omega t/b}, \quad \omega,\, t \in \mathbb{R}.\end{aligned}$$

where the corresponding parameter matrix A' is given as (see [8])

$$\begin{pmatrix} d & -b & bq - dp \\ -c & a & cp - aq \end{pmatrix}.$$

In [10], the authors introduced a generalized translation operator related to the SAFT $\mathscr{F}_A$ which was called *A-translation* T_x^A with a real *shift parameter* x. For a function $f : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$ it reads as

$$T_x^A f(t) = \mathrm{e}^{-\mathrm{i}\frac{a}{b}x(t-x)}\, f(t-x), \quad t \in \mathbb{R}. \tag{2.6}$$

Obviously, T_x^A is a norm-preserving operator on all spaces $L^p(\mathbb{R})$, $1 \le p \le \infty$, and it reduces to the ordinary translation $T_x f(t) = f(t-x)$ if $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. However, note that $T_x^A(f \cdot g)$ is in general different from $(T_x^A f) \cdot (T_x^A g)$. The following statements regarding the A-translation operator T_x^A can be found in [10]. The proofs of these statements are rather straightforward and will be omitted here.

Proposition 2.1 *For the A-translation operator T_x^A and any $f \in L^1(\mathbb{R})$, the following properties hold*

(i) $T_x^A f(t) = \rho_A(x)\,\overline{\rho_A(t)}\,\rho_A(t-x)\, f(t-x)$,
(ii) $T_x^A\, T_y^A = \mathrm{e}^{-\mathrm{i}\frac{a}{b}xy}\, T_{x+y}^A, \quad x,\, y \in \mathbb{R}$,
(iii) $\mathscr{F}_A(T_x^A f)(\omega) = \rho_A(x)\,\mathrm{e}^{-\mathrm{i}\omega x/b}\,\mathscr{F}_A f(\omega), \quad x,\, \omega \in \mathbb{R}$.

3 Shannon Sampling Series for SAFT

In [6, 10, 25, 31], an analogue of the Shannon sampling theorem for the SAFT $\mathscr{F}_A$ was obtained. In order to formulate this sampling theorem, let us introduce some notations. For given $L > 0$, a function $f \in L^2(\mathbb{R})$ is called *$\mathscr{F}_A$-bandlimited in* $[-\pi L\,|b|, \pi L\,|b|]$, if

$$\operatorname{supp}(\mathscr{F}_A f) \subseteq [-\pi L\,|b|, \pi L\,|b|].$$

For $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, we obtain the classical notion of a bandlimited function, viz. $f \in L^2(\mathbb{R})$ is *$\mathscr{F}$-bandlimited in* $[-\pi L, \pi L]$, if $\operatorname{supp}(\mathscr{F} f) \subseteq [-\pi L, \pi L]$. Note that $L > 0$ is the so-called *sampling density*. There is a close relation between $\mathscr{F}_A$-bandlimited and $\mathscr{F}$-bandlimited functions. More precisely we have

Proposition 3.1 *Let $L > 0$ be given. A function $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in the interval $[-\pi L\,|b|, \pi L\,|b|]$ if and only if the* associated function $f_A = \rho_A\, f \in L^2(\mathbb{R})$ *is $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$.*

Proof Assume that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Using (2.3), we see immediately that

$$\operatorname{supp}\big(\mathscr{F}(\rho_A\, f)\big) \subseteq [-\pi L, \pi L],$$

i.e., $f_A = \rho_A\, f$ is $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$.

Conversely, let $g \in L^2(\mathbb{R})$ be $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$. Then from (2.3) it follows that

$$(\mathscr{F} g)(\omega) = \sqrt{|b|}\; \overline{\eta_A(\omega)}\, \mathscr{F}_A(\bar{\rho}_A\, g)(b\,\omega), \quad \omega \in \mathbb{R}.$$

Hence $\bar{\rho}_A\, g \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. □

Remark 3.2 The equivalence class of functions $f \in L^2(\mathbb{R})$ with $\operatorname{supp}\big(\mathscr{F}(\rho_A\, f)\big) \subseteq [-\pi L, \pi L]$ always contains a smooth function. Indeed, for any $r \in \mathbb{N}_0$ the function $(\mathrm{i}\omega)^r\, \mathscr{F}(\rho_A\, f)(\omega)$ is in $L^1([-\pi L, \pi L])$ such that

$$(\rho_A\, f)^{(r)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi L}^{\pi L} \mathscr{F}(\rho_A\, f)(\omega)\, (\mathrm{i}\omega)^r\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega$$

belongs to $C_0(\mathbb{R})$. Clearly, with $\rho_A\, f$ also f is smooth. In the sequel, we will always pick the smooth representative of an $\mathscr{F}_A$-bandlimited function.

Example 3.3 The *cardinal sine function* is defined as usual by

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \in \mathbb{R} \setminus \{0\}, \\ 1, & t = 0. \end{cases} \tag{3.1}$$

For fixed $L > 0$, the function $\operatorname{sinc}(L\,\cdot)$ is $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$. Thus by Proposition 3.1, the function

$$\psi(t) = \overline{\rho_A(t)}\, \operatorname{sinc}(Lt), \quad t \in \mathbb{R}, \tag{3.2}$$

is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$, and by Proposition 2.1 (i) we have

$$T_x^A \psi(t) = \overline{\rho_A(t)}\, \rho_A(x)\, \operatorname{sinc}(L(t-x)).$$

for any $x \in \mathbb{R}$.

The statements of the next lemma might be well-known. We therefore omit its proof.

Lemma 3.4 *The set of shifted cardinal sine functions* $\operatorname{sinc}(\cdot - n)$ *with* $n \in \mathbb{Z}$ *forms an orthonormal system in* $L^2(\mathbb{R})$*, i.e.,*

$$\int_{\mathbb{R}} \operatorname{sinc}(\omega - m)\operatorname{sinc}(\omega - n)\,\mathrm{d}\omega = \delta_{m,n}, \quad m,\,n \in \mathbb{Z},$$

with the Kronecker symbol $\delta_{m,n}$*. For each* $x \in \mathbb{R}$*, the following identities hold*

$$\sum_{n\in\mathbb{Z}} \operatorname{sinc}(x-n) = 1, \quad \sum_{n\in\mathbb{Z}} \left|\operatorname{sinc}(x-n)\right|^2 = 1. \tag{3.3}$$

It is an easy consequence that the system $\{\sqrt{L}\operatorname{sinc}(L\cdot - n) : n \in \mathbb{Z}\}$ is an orthonormal basis for the space of functions in $L^2(\mathbb{R})$ which are $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$. The following result is the SAFT analogue of the above statement.

Lemma 3.5 *Let* $\psi(t) = \overline{\rho_A(t)}\operatorname{sinc}(Lt)$ *with* $L > 0$*. Then* $\left\{\sqrt{L}\,T^A_{n/L}\psi \,:\, n \in \mathbb{Z}\right\}$ *is an orthonormal basis for the subspace of* $L^2(\mathbb{R})$ *of all functions which are* $\mathscr{F}_A$*-bandlimited in* $[-\pi L|b|, \pi L|b|]$*.*

Proof Using Lemma 3.4, we obtain for all $m,\ n \in \mathbb{Z}$ that

$$\begin{aligned}\langle T^A_{m/L}\psi,\, T^A_{n/L}\psi\rangle &= \rho_A\left(\frac{m}{L}\right)\overline{\rho_A\left(\frac{n}{L}\right)}\,\langle \operatorname{sinc}(L\,\cdot - m), \operatorname{sinc}(L\,\cdot - n)\rangle \\ &= \frac{1}{L}\,\rho_A\left(\frac{m}{L}\right)\overline{\rho_A\left(\frac{n}{L}\right)}\,\langle \operatorname{sinc}(\cdot - m),\, \operatorname{sinc}(\cdot - n)\rangle = \frac{1}{L}\,\delta_{m,n}.\end{aligned}$$

Hence $\left\{\sqrt{L}\,T^A_{n/L}\psi \,:\, n \in \mathbb{Z}\right\}$ is an orthonormal system. It can be easily shown that the system is complete in the space of functions which are $\mathscr{F}_A$-bandlimited in $[-\pi L|b|, \pi L|b|]$. □

The following Shannon sampling theorem for the SAFT was proved in [7, 10, 25, 31]. Using Proposition 3.1, the sampling theorem for the SAFT is simple consequence of the classical Shannon sampling theorem for $\mathscr{F}$-bandlimited functions.

Theorem 3.6 *Let* $f \in L^2(\mathbb{R})$ *be* $\mathscr{F}_A$*-bandlimited in* $[-\pi L\,|b|, \pi L\,|b|]$ *and let* $f_A = \rho_A\, f$ *be the associated function of* f*. Then*

$$\begin{aligned} f(t) &= \overline{\rho_A(t)}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{L}\right)\rho_A\left(\frac{n}{L}\right)\operatorname{sinc}(Lt-n) \\ &= \overline{\rho_A(t)}\sum_{n\in\mathbb{Z}} f_A\left(\frac{n}{L}\right)\operatorname{sinc}(Lt-n) \end{aligned} \tag{3.4}$$

for every $t \in \mathbb{R}$*, where the series (3.4) converges absolutely and uniformly on* $\mathbb{R}$*. Furthermore,*

$$\sum_{n\in\mathbb{Z}} \left|f\left(\frac{n}{L}\right)\right|\,|\operatorname{sinc}(Lt-n)| \le \sqrt{L}\,\|f\|_2 \tag{3.5}$$

for all $t \in \mathbb{R}$.

Proof By assumption, $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Then by Proposition 3.1, the associated function $f_A = \rho_A f$ is $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$. Applying the classical Shannon sampling theorem to the $\mathscr{F}$- bandlimited function f_A, we obtain

$$f_A(t) = \sum_{n\in\mathbb{Z}} f_A\Big(\frac{n}{L}\Big)\,\mathrm{sinc}(Lt-n)$$

for all $t \in \mathbb{R}$, where the above series converges absolutely and uniformly on $\mathbb{R}$. This implies that the series

$$\begin{aligned} f(t) = \overline{\rho_A(t)}\, f_A(t) &= \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f_A\Big(\frac{n}{L}\Big)\,\mathrm{sinc}(Lt-n)\\ &= \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{L}\Big)\,\rho_A\Big(\frac{n}{L}\Big)\,\mathrm{sinc}(Lt-n), \quad t \in \mathbb{R} \end{aligned}$$

converges absolutely and uniformly on $\mathbb{R}$, since $|\rho_A(t)| = 1$ for each $t \in \mathbb{R}$.

By Lemma 3.4 we have

$$\sum_{n\in\mathbb{Z}} |\mathrm{sinc}(Lt-n)|^2 = 1, \quad t \in \mathbb{R}. \tag{3.6}$$

The absolute convergence of the Shannon sampling series (3.8) for $\mathscr{F}_A$ and the estimate (3.5) can be shown by applying the Cauchy–Schwarz inequality as follows:

$$\begin{aligned} &\big|\overline{\rho_A(t)}\big| \sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|\,\Big|\rho_A\Big(\frac{n}{L}\Big)\Big|\,\Big|\mathrm{sinc}(Lt-n)\Big|\\ &= \sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|\,\big|\mathrm{sinc}(Lt-n)\big|\\ &\le \Big(\sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|^2\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}} \big|\mathrm{sinc}(Lt-n)\big|^2\Big)^{1/2}\\ &= \Big(\sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|^2\Big)^{1/2}. \end{aligned}$$

By Lemma 3.4 and (3.4) it follows that

$$\|f\|_2^2 = \langle f, f\rangle = \sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|^2 \int_{\mathbb{R}} \big(\mathrm{sinc}(Lt-n)\big)^2 \mathrm{d}t = \frac{1}{L} \sum_{n\in\mathbb{Z}} \Big|f\Big(\frac{n}{L}\Big)\Big|^2. \tag{3.7}$$

This completes the proof. □

An equivalent formulation of Theorem 3.6 can be given in terms of the A-translates $T^A_{n/L}$ of the function ψ defined in Example 3.3.

Corollary 3.7 *Let $\psi(t) = \overline{\rho_A(t)}\,\mathrm{sinc}(Lt)$ with $L > 0$ be given. If $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$, then*

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{L}\right)(T^A_{n/L}\psi)(t) \tag{3.8}$$

for every $t \in \mathbb{R}$, where the series (3.8) converges absolutely and uniformly on $\mathbb{R}$.

Proof By Example 3.3, the function ψ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Using the A-translation (2.6) with shift parameter $x = \frac{n}{L}$, we obtain for all $n \in \mathbb{Z}$,

$$(T^A_{n/L}\psi)(t) = T^A_{n/L}\big(\bar{\rho}_A\,\mathrm{sinc}(L\,\cdot)\big)(t) = \overline{\rho_A(t)}\,\rho_A\Big(\frac{n}{L}\Big)\,\mathrm{sinc}(Lt - n) \tag{3.9}$$

such that from Theorem 3.6 the representation (3.8) of f *follows*. □

By Lemma 3.4, the system $\{T_{n/L}\psi : n \in \mathbb{Z}\}$ is an orthonormal basis for the space of functions which are $\mathscr{F}_A$-bandlimited in $[-\pi L|b|, \pi L|b|]$, thus convergence of (3.8) resp. (3.4) also holds with respect to the $\|\cdot\|_2$-norm.

Remark 3.8 For some special *SAFT*'s, corresponding Shannon sampling theorems were studied mainly in the signal processing literature. For $A = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with $b \neq 0$, Theorem 3.6 implies the sampling theorem of the Fresnel transform $\mathscr{F}_A$ (see [12]). For $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \end{pmatrix}$ with $\theta \notin \pi\mathbb{Z}$, Theorem 3.6 yields the sampling theorem for the fractional Fourier transform $\mathscr{F}_A$ (see [5, 6, 33]).

For the parameter matrix $A = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 \end{pmatrix}$ with $\theta \neq 0$, Theorem 3.6 yields a sampling theorem for the hyperbolic transform. For $A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \in \mathbb{R}^{2\times 3}$ with $ad - bc = 1$ and $abd \neq 0$, Theorem 3.6 implies the sampling theorem of the linear canonical transform $\mathscr{F}_A$ (see [24, 29, 34]), which reads for $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\mathscr{F}_A f) \subseteq [-\pi L\,|b|, \pi L\,|b|]$ as follows

$$f(t) = \mathrm{e}^{-\mathrm{i}\frac{a}{2b}t^2} \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{L}\Big)\,\mathrm{e}^{\mathrm{i}\frac{a}{2b}n^2/L^2}\,\mathrm{sinc}(Lt - n)$$

for all $t \in \mathbb{R}$.

Suppose that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ which, without loss of generality, can be assumed to be smooth (see Remark 3.2). Although the reconstruction of f from samples $f\left(\frac{n}{L}\right)$, $n \in \mathbb{Z}$, is possible according to Theorem

3.6, it is numerically an unstable process in the deterministic sense. Consider for a sufficiently large $N \in \mathbb{N}$ erroneous samples of $f\left(\frac{n}{L}\right)$, $n \in \mathbb{Z}$, given as

$$\tilde{f}\left(\frac{n}{L}\right) = \begin{cases} f\left(\frac{n}{L}\right) + \varepsilon_n & n \in \{-N, \dots, N\}, \\ f\left(\frac{n}{L}\right) & n \in \mathbb{Z} \setminus \{-N, \dots, N\} \end{cases}$$

with complex error terms ε_n which are uniformly bounded by $|\varepsilon_n| \le \varepsilon$. Then we obtain the function

$$\begin{aligned} g(t) &= \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} \tilde{f}\left(\frac{n}{L}\right) \rho_A\left(\frac{n}{L}\right) \operatorname{sinc}(Lt - n) \\ &= f(t) + \overline{\rho_A(t)} \sum_{n=-N}^{N} \varepsilon_n \, \rho_A\left(\frac{n}{L}\right) \operatorname{sinc}(Lt - n), \quad t \in \mathbb{R}. \end{aligned}$$

An upper estimate for the sup-norm of the difference of f and g is given by

$$\|g - f\|_\infty \le \varepsilon \max_{t\in\mathbb{R}} \sum_{n=-N}^{N} |\sin(Lt - n)|.$$

By [14, Theorem 2.2] we have

$$\max_{t\in\mathbb{R}} \sum_{n=-N}^{N} |\operatorname{sinc}(Lt - n)| < \frac{2}{\pi} \left(\ln(N) + 2\ln(2) + \gamma\right) + \frac{N+2}{\pi N(N+1)},$$

with *Euler's constant* [32]

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln(N)\right) = 0.57721566\dots.$$

Now we shall give a lower bound for the approximation of f utilizing perturbed sample values. The result shows in particular that the reconstruction of a $\mathscr{F}_A$-bandlimited function f by the Shannon sampling series (3.8) for $\mathscr{F}_A$ is numerically unstable in the deterministic sense.

Theorem 3.9 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with fixed $L > 0$. For arbitrary $N \in \mathbb{N}$ and $\varepsilon > 0$ define*

$$\varepsilon_n = \varepsilon \operatorname{sign}\big(\operatorname{sinc}(\tfrac{1}{2} - n)\big)\, \overline{\rho_A\left(\frac{n}{L}\right)} = \varepsilon\,(-1)^{n+1} \operatorname{sign}(2n - 1)\, \overline{\rho_A\left(\frac{n}{L}\right)}$$

for $|n| \le N$ and $\varepsilon_n = 0$ for $|n| > N$. Then

$$\|g - f\|_\infty \ge \varepsilon \left(\frac{2}{\pi} \ln(N) + \frac{4}{\pi} \ln(2) + \frac{2\gamma}{\pi}\right) > \varepsilon\left(\frac{2}{\pi} \ln(N) + \frac{5}{4}\right).$$

Proof The special choice of the complex error terms ε_n leads to

$$g(t) - f(t) = \varepsilon\, \overline{\rho_A(t)} \sum_{n=-N}^{N} \operatorname{sign}\Big(\operatorname{sinc}(\tfrac{1}{2} - n)\Big) \operatorname{sinc}(Lt - n).$$

For $t = \frac{1}{2L}$ we obtain

$$\begin{aligned}\|g - f\|_\infty \geq |g(\tfrac{1}{2L}) - f(\tfrac{1}{2L})| &= \varepsilon \sum_{k=-N}^{N} \big|\operatorname{sinc}(\frac{1}{2} - n)\big| \\ &= \frac{2\varepsilon}{\pi} + \frac{2\varepsilon}{\pi} \sum_{n=1}^{N} \Big(\frac{1}{2n-1} + \frac{1}{2n+1}\Big) \\ &= \frac{2\varepsilon}{(2N+1)\pi} + \frac{4\varepsilon}{\pi} \sum_{n=1}^{N} \frac{1}{2n-1}.\end{aligned}$$

As shown in [14, Formula (2.8)], it holds

$$\sum_{n=1}^{N} \frac{1}{2n-1} > \frac{1}{2} \ln(N) + \ln(2) + \frac{\gamma}{2} - \frac{1}{4N(2N+1)}.$$

Hence this yields the estimate

$$\begin{aligned}\|\tilde{f} - f\|_\infty &> \varepsilon \Big(\frac{2}{\pi} \ln(N) + \frac{4}{\pi} \ln(2) + \frac{2\gamma}{\pi}\Big) + \varepsilon\, \frac{2N-1}{\pi\, N(2N+1)} \\ &> \varepsilon \Big(\frac{2}{\pi} \ln(N) + \frac{4}{\pi} \ln(2) + \frac{2\gamma}{\pi}\Big).\end{aligned}$$

Since $\frac{4}{\pi} \ln(2) + \frac{2\gamma}{\pi} = 1.2500093\ldots > \frac{5}{4}$, we get the final estimate. □

Remark 3.10 Assume that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. Let $t \in \mathbb{R}$ be arbitrary fixed. In Theorem 3.9 we have seen that the approximation of f by the N-th partial sum f_N of the corresponding Shannon sampling series for $\mathscr{F}_A$, which is given by

$$f_N(t) = \overline{\rho_A(t)} \sum_{n=-N}^{N} f\Big(\frac{n}{L}\Big)\, \rho_A\Big(\frac{n}{L}\Big) \operatorname{sinc}(Lt - n),$$

is not numerically robust in the deterministic sense. Otherwise, a simple average case study (see [30]) shows that this approximation is numerically robust in the stochastic sense. For this we suppose that instead of the exact samples $f\left(\frac{n}{L}\right)$ only noisy samples

$\tilde{f}\left(\frac{n}{L}\right) = f\left(\frac{n}{L}\right) + X_n$, $n = -N, \ldots, N$, are given, where the complex random variables X_n are uncorrelated, each of them having expectation $\mathbb{E}(X_n) = 0$ and constant variance $\mathbb{V}(X_n) = \mathbb{E}\big(|X_n|^2\big) = \rho^2$ with some $\rho > 0$. Then we form the function

$$g_N(t) = \overline{\rho_A(t)} \sum_{n=-N}^{N} \tilde{f}\Big(\frac{n}{L}\Big)\, \rho_A\Big(\frac{n}{L}\Big)\, \mathrm{sinc}(Lt - n)$$

and consider the stochastic approximation error

$$\Delta_N(t) = g_N(t) - f_N(t) = \overline{\rho_A(t)} \sum_{n=-N}^{N} X_n\, \rho_A\Big(\frac{n}{L}\Big)\, \mathrm{sinc}(Lt - n).$$

Obviously, this stochastic error $\Delta_N(t)$ has the expectation

$$\mathbb{E}\big(\Delta_N(t)\big) = \overline{\rho_A(t)} \sum_{n=-N}^{N} \mathbb{E}\big(X_n\big)\, \rho_A\Big(\frac{n}{L}\Big)\, \mathrm{sinc}(Lt - n) = 0$$

and the variance

$$\begin{aligned}\mathbb{V}\big(\Delta_N(t)\big) &= \big|\overline{\rho_A(t)}\big|^2 \sum_{n=-N}^{N} \mathbb{V}\big(X_n\big)\, \Big|\rho_A\Big(\frac{n}{L}\Big)\Big|^2\, \big|\mathrm{sinc}(Lt - n)\big|^2 \\ &= \rho^2 \sum_{n=-N}^{N} \big|\mathrm{sinc}(Lt - n)\big|^2.\end{aligned}$$

From (3.4) it follows that

$$\sum_{n=-N}^{N} \big|\mathrm{sinc}(Lt - n)\big|^2 \le 1$$

such that $\mathbb{V}\big(\Delta_N(t)\big) \le \rho^2$.

4 Regularized Shannon Sampling Formulas for SAFT

In this section, we assume that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with some $L > 0$. Then f is also $\mathscr{F}_A$-bandlimited in $[-\pi T\,|b|, \pi T\,|b|]$, where $T > L$. Instead of L, we use T as sampling density. Our aim is to improve the representation of f by the Shannon sampling series for $\mathscr{F}_A$ with sampling density T, i.e.,

$$f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{T}\Big)\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n), \quad t \in \mathbb{R}. \tag{4.1}$$

The practical use of the Shannon sampling series (4.1) for $\mathscr{F}_A$ is rather limited due to the need of infinitely many samples. A second limitation of its use in practice originates in the fact that the scaled cardinal sine function $\mathrm{sinc}(T\,\cdot)$ decays very slowly, which results in poor convergence of the Shannon sampling series. Furthermore, in case of noisy samples $\tilde{f}\left(\frac{n}{T}\right)$, $n \in \mathbb{Z}$, the convergence of the Shannon sampling series for $\mathscr{F}_A$ can even break down completely (see [11, 15] and Theorem 3.9). In order to overcome these limitations, regularization techniques were considered. For the classical Fourier transform, the concept of regularized Shannon sampling formulas with localized sampling and oversampling has been studied by several authors [14, 15, 17, 21–23, 28]. Often a Gaussian window function supported on whole $\mathbb{R}$ was used in these studies. In [14], it was shown that the compactly supported sinh-type window function (4.3) produces smaller approximation errors than the Gaussian window function. Therefore we focus on compactly supported, continuous window functions in the following.

For any $m \in \mathbb{N} \setminus \{1\}$ let $\Phi_{m/T}$ be the class of even continuous functions $\varphi : \mathbb{R} \to [0, 1]$ with the following properties:

(i) φ is supported on $\left[-\frac{m}{T}, \frac{m}{T}\right]$,

(ii) φ is monotonically decreasing on $\left[0, \frac{m}{T}\right]$ with $\varphi(0) = 1$ and $\varphi\left(\frac{m}{T}\right) = 0$.

A function of the class $\Phi_{m/T}$ will henceforth be called *window function* with the *truncation parameter* m, where $T > L$ denotes the sampling density. Later we will use one of the following window functions of $\Phi_{m/T}$.

Example 4.1 (a) Let M_{2s} denote the centered cardinal B-spline of even order $2s \in 2\mathbb{N}$. The B-*spline window function* is defined as

$$\varphi_{\mathrm{B}}(t) = \frac{1}{M_{2s}(0)}\, M_{2s}\Big(\frac{T\,s\,t}{m}\Big), \quad t \in \mathbb{R}. \tag{4.2}$$

(b) The sinh-*type window function* is defined for the parameter $\beta = \frac{m\pi\,(T-L)}{T}$ as

$$\varphi_{\sinh}(t) = \begin{cases} \frac{1}{\sinh \beta}\, \sinh\left(\beta\sqrt{1 - (\frac{T t}{m})^2}\right) & t \in \left[-\frac{m}{T}, \frac{m}{T}\right], \\ 0 & t \in \mathbb{R} \setminus \left[-\frac{m}{T}, \frac{m}{T}\right]. \end{cases} \tag{4.3}$$

(c) Let again $\beta = \frac{m\pi\,(T-L)}{T}$. The *continuous Kaiser–Bessel window function* is defined as

$$\varphi_{\mathrm{cKB}}(t) = \begin{cases} \frac{1}{I_0(\beta)-1}\left(I_0\left(\beta\sqrt{1 - (\frac{T t}{m})^2}\right) - 1\right) & t \in \left[-\frac{m}{T}, \frac{m}{T}\right], \\ 0 & t \in \mathbb{R} \setminus \left[-\frac{m}{T}, \frac{m}{T}\right], \end{cases} \tag{4.4}$$

where I_0 is the *modified Bessel function of the first kind*, given by

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{((2k)!!)^2}\, x^{2k}, \quad x \in \mathbb{R}.$$

We will consider the following regularization strategies for Shannon sampling series related to $\mathscr{F}_A$:

1. A better approximation of a function $f \in L^2(\mathbb{R})$ which is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ can be obtained by so-called *oversampling* with a sampling density $T > L$, i.e., instead of the samples $f\left(\frac{n}{L}\right)$, $n \in \mathbb{Z}$, we work now with the samples $f\left(\frac{n}{T}\right)$, $n \in \mathbb{Z}$. Oversampling means that we apply a larger sampling density $T > L$ for the reconstruction of f.
2. The Shannon sampling series (4.1) for $\mathscr{F}_A$ will be regularized by a window function $\varphi \in \Phi_{m/T}$ with moderate $m \in \mathbb{N} \setminus \{1\}$. Thus we consider instead of the series

$$\overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt-n)$$

the *regularized Shannon sampling formula for* $\mathscr{F}_A$ with sampling density T given by

$$R^A_{\varphi,m} f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt-n)\, \varphi\left(t-\frac{n}{T}\right), \quad t \in \mathbb{R}. \tag{4.5}$$

Since $\varphi \in \Phi_{m/T}$ is compactly supported, the computation of $R^A_{\varphi,m} f(t)$ for fixed $t \in \mathbb{R}$ needs only a finite number of samples of f.

Note that since

$$\overline{\rho_A(t)}\, \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt-n)\, \varphi\left(t-\frac{n}{T}\right)\Big|_{t=k/T} = \delta_{n,k}, \quad n, k \in \mathbb{Z},$$

we have an *interpolating approximation* of f on the grid $\frac{1}{T}\mathbb{Z}$, viz.

$$R^A_{\varphi,m} f\left(\frac{k}{T}\right) = f\left(\frac{k}{T}\right), \quad k \in \mathbb{Z}.$$

Furthermore, the use of a window function $\varphi \in \Phi_{m/T}$ implies that the computation of $R^A_{\varphi,m} f(t)$ for $t \in \mathbb{R} \setminus \frac{1}{T}\mathbb{Z}$ requires only $2m$ samples $f\left(\frac{n}{T}\right)$, where $n \in \mathbb{Z}$ fulfills

$|n - Tt| < Tm$. Hence the function f can be recovered on the interval $\left[0,\, \frac{1}{T}\right]$ by

$$f(t) = \begin{cases} f(0) & t = 0, \\ f\Big(\frac{1}{T}\Big) & t = \frac{1}{T}, \\ \overline{\rho_A(t)} \displaystyle\sum_{n=1-m}^{m} f\Big(\frac{n}{T}\Big)\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n)\, \varphi\Big(t - \frac{n}{T}\Big) & t \in \Big(0,\, \frac{1}{T}\Big). \end{cases}$$

Thus the reconstruction of f on the interval $[-1, 1]$ needs only $2T + 2m - 1$ samples $f\Big(\frac{n}{T}\Big)$ with $n = -T - m, \ldots, T + m$. This fact is called *localized sampling* of f.

Remark 4.2 By (4.5), the expression $R^A_{\varphi,m} f$ is a linear combination of A-translates of the function $\chi(t) = \overline{\rho_A(t)}\, \mathrm{sinc}(Tt)\, \varphi(t)$. Using (2.6) we have

$$T^A_{n/T}\chi(t) = \overline{\rho_A(t)}\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n)\, \varphi\Big(t - \frac{n}{T}\Big), \quad n \in \mathbb{Z},$$

and hence

$$R^A_{\varphi,m} f(t) = \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{T}\Big)\, T^A_{n/T}\chi(t).$$

Note that since φ is a function from the class $\Phi_{m/T}$ it has support $\left[-\frac{m}{T},\, \frac{m}{T}\right]$ and this implies that for fixed $t \in \mathbb{R}$ the representation $R^A_{\varphi,m} f(t)$ is a finite sum.

Now we give an estimate for the uniform approximation error $\|f - R^A_{\varphi,m} f\|_\infty$, where $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$.

Theorem 4.3 *Let $0 < L < T$ and $m \in \mathbb{N}\backslash\{1\}$ be given. Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Further let $\varphi \in \Phi_{m/T}$ be a given window function.*

Then the error of the regularized Shannon sampling formula (4.5) of $\mathscr{F}_A$ satisfies the estimate

$$\|f - R^A_{\varphi,m} f\|_\infty \le E(m, L, T)\, \|f\|_2$$

with the error constant

$$E(m, L, T) = \sqrt{L} \max_{\omega\in[-\pi L, \pi L]} \Big|1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi T}^{\omega+\pi T} (\mathscr{F}\varphi)(\tau)\, \mathrm{d}\tau\Big|. \tag{4.6}$$

Proof By assumption, $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Then by Proposition 3.1, the associated function $f_A = \rho_A f$ is $\mathscr{F}$-bandlimited in $[-\pi L, \pi L]$.

We choose $T > L$ as sampling density and apply the regularized Shannon sampling formula for f_A related to the classical Fourier transform $\mathscr{F}$, i.e.,

$$R_{\varphi,m} f_A(t) = \sum_{n\in\mathbb{Z}} f_A\Big(\frac{n}{T}\Big)\,\mathrm{sinc}(Tt-n)\,\varphi\Big(t-\frac{n}{T}\Big)$$
$$= \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{T}\Big)\,\rho_A\Big(\frac{n}{T}\Big)\,\mathrm{sinc}(Tt-n)\,\varphi\Big(t-\frac{n}{T}\Big).$$

As shown in [14, Theorem 3.2], it holds the error estimate

$$\|f_A - R^A_{\varphi,m} f_A\|_\infty \le E(m, L, T)\,\|f_A\|_2$$

with the error constant (4.6). Note that in [14] there was used a different form of the Fourier transform. From $f = \bar\rho_A\, f_A$ and $R^A_{\varphi,m} f = \bar\rho_A\, R_{\varphi,m}\, f_A$ it follows immediately that

$$\|f - R^A_{\varphi,m} f\|_\infty = 1\cdot\|f_A - R_{\varphi,m} f_A\|_\infty, \quad \|f\|_2 = 1\cdot\|f_A\|_2.$$

This completes the proof. □

Remark 4.4 The error constant (4.6) is independent of f and $\mathscr{F}_A$. It measures the regularisation effect of $\varphi \in \Phi_{m/T}$ by oversampling with sampling density $T > L$. For each $\omega \in [-\pi L, \pi L]$ we have $\omega - \pi T < 0$ and $\omega + \pi T > 0$. By $\varphi \in \Phi_{m/T}$ it holds

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathscr{F}\varphi)(\omega)\,\mathrm{d}\omega = (\mathscr{F}^{-1}\,\mathscr{F}\varphi)(0) = \varphi(0) = 1.$$

For a suitable window function $\varphi \in \Phi_{m/T}$ with moderate $m \in \mathbb{N}\setminus\{1\}$, the error constant $E(m, L, T)$ can be sufficiently small as we will demonstrate in Sect. 5.

In Theorem 3.9, we have seen that the Shannon sampling series for $\mathscr{F}_A$ does not behave stably with respect to perturbed samples of an $\mathscr{F}_A$-bandlimited function f. Now we are considering this problem for the regularized Shannon sampling formula related to $\mathscr{F}_A$. It turns out that in contrast to the Shannon sampling series (3.8) the regularized Shannon sampling formula (4.5) for $\mathscr{F}_A$ is numerically robust, i.e., the uniform error is small for perturbed samples of an $\mathscr{F}_A$-bandlimited function f. More precisely we have the following statement.

Theorem 4.5 *Let $0 < L < T$ and $m \in \mathbb{N}\setminus\{1\}$ be given. Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$. Further let $\varphi \in \Phi_{m/T}$ be a given window function. Furthermore, let $\tilde f\big(\frac{n}{T}\big) = f\big(\frac{n}{T}\big) + \varepsilon_n$, $n \in \mathbb{Z}$, be perturbed samples with complex error terms ε_n which are uniformly bounded by $|\varepsilon_n| \le \varepsilon$ with $0 < \varepsilon \ll 1$. Let*

$$h(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} \tilde f\Big(\frac{n}{T}\Big)\,\rho_A\Big(\frac{n}{T}\Big)\,\mathrm{sinc}(Tt-n)\,\varphi\Big(t-\frac{n}{T}\Big), \quad t \in \mathbb{R}.$$

Then the following estimates hold

$$\|h - R^A_{\varphi,m} f\|_\infty \le \varepsilon \left(2 + \sqrt{2\pi}\, \mathscr{F}\varphi(0)\right), \tag{4.7}$$

$$\|f - h\|_\infty \le \|f - R^A_{\varphi,m} f\|_\infty + \varepsilon \left(2 + \sqrt{2\pi}\, \mathscr{F}\varphi(0)\right). \tag{4.8}$$

Proof We consider the perturbation error

$$e(t) = h(t) - R^A_{\varphi,m} f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} \varepsilon_n\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n)\, \varphi\Big(t - \frac{n}{T}\Big).$$

Note that for any $t \in \mathbb{R}$ the above sum contains only finitely many non-vanishing terms. First we consider the case $t \in \left(0, \frac{1}{T}\right)$. Due to the properties of $\varphi \in \Phi_{m/T}$ and the fact $|\varepsilon_n| \le \varepsilon$, we obtain

$$|e(t)| \le \sum_{n=1-m}^{m} |\varepsilon_n|\, |\mathrm{sinc}(Tt - n)|\, \varphi\Big(t - \frac{n}{T}\Big) \le \varepsilon \sum_{n=1-m}^{m} \varphi\Big(t - \frac{n}{T}\Big).$$

Since $\varphi \in \Phi_m$ is monotonically decreasing on $\left[0, \frac{m}{T}\right]$, we have for $t \in \left(0, \frac{1}{T}\right)$ that

$$\begin{aligned}\sum_{n=1-m}^{m} \varphi\Big(t - \frac{n}{T}\Big) &= \Big(\sum_{n=1-m}^{0} + \sum_{n=1}^{m}\Big)\varphi\Big(t - \frac{n}{T}\Big) = \sum_{n=0}^{m-1} \varphi\Big(t + \frac{n}{T}\Big) + \sum_{n=1}^{m} \varphi\Big(t - \frac{n}{T}\Big) \\ &\le \sum_{n=0}^{m-1} \varphi\Big(\frac{n}{T}\Big) + \sum_{n=1}^{m} \varphi\Big(\frac{1-n}{T}\Big) = 2 \sum_{n=0}^{m-1} \varphi\Big(\frac{n}{T}\Big).\end{aligned}$$

The latter sum can be estimated further by applying once more the monotonicity of φ on $\left[0, \frac{m}{T}\right]$. We obtain

$$\sum_{n=0}^{m-1} \varphi\Big(\frac{n}{T}\Big) < \varphi(0) + \int_0^{(m-1)/T} \varphi(t)\, \mathrm{d}t \le 1 + \int_0^{m/T} \varphi(t)\, \mathrm{d}t.$$

Since $\mathscr{F}\varphi(0) = \sqrt{\frac{2}{\pi}} \int_0^{m/T} \varphi(t)\, \mathrm{d}t$, we eventually have

$$|e(t)| \le 2\,\varepsilon \sum_{n=0}^{m-1} \varphi\Big(\frac{n}{T}\Big) \le \varepsilon \left(2 + \sqrt{2\pi}\, \mathscr{F}\varphi(0)\right)$$

for every $t \in \left(0, \frac{1}{T}\right)$. Note that $|e(0)| = |\varepsilon_0| \le \varepsilon$ and $\left|e\left(\frac{1}{T}\right)\right| = |\varepsilon_1| \le \varepsilon$, which leads to

$$\max_{t\in[0,1/T]} |e(t)| \le \varepsilon \left(2 + \sqrt{2\pi}\, \mathscr{F}\varphi(0)\right). \tag{4.9}$$

The same technique can obviously be applied to get the estimate (4.9) for every interval $\left[\frac{k}{T}, \frac{k+1}{T}\right]$, $k \in \mathbb{Z}$, and thus (4.7) follows.

The inequality (4.8) can easily be derived by applying the triangular inequality and (4.7). □

5 Specific Regularized Shannon Sampling Formulas for the SAFT

In this section we will study regularized Shannon sampling series with specific window functions which we have introduced in Example 4.1. We assume again that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$.

5.1 Regularization with B-spline Window Function

Let $m \in \mathbb{N}\backslash\{1\}$ and $s \in \mathbb{N}$ with $s < \frac{\pi m}{2}$ be given. A good choice is $s = \lfloor m/2 \rfloor$. Set $T > \frac{\pi m}{\pi m - 2s}\, L$. First we consider the B-spline window function $\varphi_{\mathrm{B}} \in \Phi_{m/T}$ of order $2s$ as defined in (4.2). The B-*spline regularized Shannon sampling formula for* $\mathscr{F}_A$ with sampling density T reads as follows

$$R^A_{\mathrm{B},m} f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \mathrm{sinc}(Tt - n)\, \varphi_{\mathrm{B}}\left(t - \frac{n}{T}\right). \tag{5.1}$$

We will now show that the uniform approximation error $\|f - R^A_{\mathrm{B},m} f\|_\infty$ decays exponentially. More precisely the following statement holds.

Theorem 5.1 *Let* $f \in L^2(\mathbb{R})$ *be* $\mathscr{F}_A$*-bandlimited in* $[-\pi L\,|b|, \pi L\,|b|]$ *with* $L > 0$. *Assume that* $m \in \mathbb{N} \setminus \{1\}$, $s \in \mathbb{N}$ *with* $s < \frac{\pi m}{2}$, *and* $T > \frac{\pi m}{\pi m - 2s}\, L$. *Let* $\varphi_{\mathrm{B}} \in \Phi_{m/T}$ *be the* B*-spline window function* (4.2) *of order* $2s$.

Then

$$\|f - R^A_{\mathrm{B},m} f\|_\infty \le \sqrt{L} \left(\frac{2sT}{m\pi\,(T - L)}\right)^{2s-1} \|f\|_2. \tag{5.2}$$

Proof According to Theorem 4.3 we have

$$\|f - R^A_{\mathrm{B},m} f\|_\infty \le \sqrt{L}\, \|f\|_2 \max_{\omega\in[-\pi L,\pi L]} \left|\Delta_{\mathrm{B}}(\omega)\right|$$

with the auxiliary function

$$\Delta_{\mathrm{B}}(\omega) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi T}^{\omega+\pi T} \mathscr{F}\varphi_{\mathrm{B}}(\tau)\,\mathrm{d}\tau, \quad \omega \in [-\pi L, \pi L].$$

By [19, p. 496] we have

$$\int_{\mathbb{R}} M_{2s}(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \Big(\mathrm{sinc}\,\frac{\omega}{2\pi}\Big)^{2s},$$

since the cardinal sine function is here defined by (3.9). Hence the B-spline window function (4.2) has the Fourier transform

$$\mathscr{F}\varphi_{\mathrm{B}}(\omega) = \frac{m}{\sqrt{2\pi}\,sT\,M_{2s}(0)} \Big(\mathrm{sinc}\,\frac{m\,\omega}{2\pi\,sT}\Big)^{2s}, \tag{5.3}$$

which results in

$$1 = \varphi_{\mathrm{B}}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathscr{F}\varphi_{\mathrm{B}}(\tau)\,\mathrm{d}\tau = \frac{m}{2\pi\,sT\,M_{2s}(0)} \int_{\mathbb{R}} \Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\,\mathrm{d}\tau.$$

Then the auxiliary function Δ_{B} takes the form

$$\begin{aligned}\Delta_{\mathrm{B}}(\omega) &= \frac{m}{2\pi\,sT\,M_{2s}(0)}\Big(\int_{\mathbb{R}}\Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\mathrm{d}\tau - \int_{\omega-\pi T}^{\omega+\pi T}\Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\mathrm{d}\tau\Big)\\ &= \frac{m}{2\pi\,sT\,M_{2s}(0)}\Big(\int_{\pi T-\omega}^{\infty}\Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\mathrm{d}\tau + \int_{\pi T+\omega}^{\infty}\Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\mathrm{d}\tau\Big).\end{aligned}$$

The single integral terms can be estimated by

$$\int_{\pi T\pm\omega}^{\infty}\Big(\mathrm{sinc}\,\frac{m\,\tau}{2\pi\,sT}\Big)^{2s}\mathrm{d}\tau \le \frac{(2sT)^{2s}}{m^{2s}}\int_{\pi T\pm\omega}^{\infty}\tau^{-2s}\,\mathrm{d}\tau = \frac{(2sT)^{2s}}{(2s-1)\,m^{2s}\,(\pi T\pm\omega)^{2s-1}}.$$

Hence we obtain for all $\omega \in [-\pi\,L, \pi\,L]$ that

$$0 \le \Delta_{\mathrm{B}}(\omega) \le \frac{(2sT)^{2s-1}}{(2s-1)\,m^{2s-1}\,\pi\,M_{2s}(0)}\Big(\frac{1}{(\pi T-\omega)^{2s-1}} + \frac{1}{(\pi T+\omega)^{2s-1}}\Big).$$

From $T > L$ and $\omega \in [-\pi L, \pi L]$ it follows that $\pi T \pm \omega \in [(T-L)\pi,\ (T+L)\pi]$. Taking into account that the function x^{1-2s} is decreasing for $x > 0$, we conclude

$$\max_{\omega\in[-\pi L,\,\pi L]}\big|\Delta_{\mathrm{B}}(\omega)\big| \le \frac{2\,(2sT)^{2s-1}}{(2s-1)\,m^{2s-1}\,\pi^{2s}\,M_{2s}(0)\,(T-L)^{2s-1}}.$$

Applying [14, Formula (5.3)] gives

$$\frac{4}{3} \le \sqrt{2s}\,M_{2s}(0) < \sqrt{\frac{6}{\pi}}. \tag{5.4}$$

Thus for the error constant (4.6) we obtain the estimate

$$E(m, L, T) \leq \frac{3}{2\pi} \sqrt{L} \left(\frac{2sT}{m\pi\,(T-L)}\right)^{2s-1} \frac{\sqrt{2s}}{2s-1} \leq \sqrt{L} \left(\frac{2sT}{m\pi\,(T-L)}\right)^{2s-1},$$

because it holds

$$\frac{3}{2\pi}\,\frac{\sqrt{2s}}{2s-1} < 1, \quad s \in \mathbb{N}.$$

By the oversampling condition $T > \frac{\pi m}{\pi m - 2s} L$ we have $0 < \frac{2sT}{m\pi\,(T-L)} < 1$. This completes the proof. □

Let $m \in \mathbb{N}\backslash\{1\}$ and $s \in \mathbb{N}$ with $s < \frac{\pi m}{2}$ be given. Now we show that for the B-spline regularized Shannon sampling formula (5.1) for $\mathscr{F}_A$, the uniform perturbation error $\|R_{\mathrm{B},m}^A f - R_{\mathrm{B},m}^A \tilde{f}\|_\infty$ is relatively small.

Theorem 5.2 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. Assume that $m \in \mathbb{N}\backslash\{1\}$, $s \in \mathbb{N}$ with $s < \frac{\pi m}{2}$, and $T \geq L$ are given. Let $\varphi_{\mathrm{B}} \in \Phi_{m/T}$ be the* B*-spline window function* (4.2) *of order* $2s$*. Furthermore, let $\tilde{f}\left(\frac{n}{T}\right) = f\left(\frac{n}{T}\right) + \varepsilon_n$, $n \in \mathbb{Z}$, be noisy samples with complex error terms ε_n which are uniformly bounded by $|\varepsilon_n| \leq \varepsilon$ with $0 < \varepsilon \ll 1$.*

Then the B*-spline regularized Shannon sampling formula* (5.1) *for $\mathscr{F}_A$ with density T is numerically robust and it holds*

$$\|R_{\mathrm{B},m}^A f - h\|_\infty \leq \varepsilon \left(2 + \frac{3\sqrt{2}\,m}{4\sqrt{s}\,T}\right),$$

where h is given by

$$h(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} \tilde{f}\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt - n)\, \varphi_{\mathrm{B}}\left(t - \frac{n}{T}\right).$$

Proof By Theorem 4.5 we only have to determine the value $\mathscr{F}\varphi_{\mathrm{B}}(0)$ for the B-spline window function (4.2). From (5.3) and (5.4) it follows that

$$\mathscr{F}\varphi_{\mathrm{B}}(0) = \frac{m}{\sqrt{2\pi}\,sT\,M_{2s}(0)} \leq \frac{3\,m}{4T\,\sqrt{s\,\pi}}$$

and hence

$$\sqrt{2\pi}\,\mathscr{F}\varphi_{\mathrm{B}}(0) \leq \frac{3\sqrt{2}\,m}{4T\,\sqrt{s}}.$$

□

If $s \in \mathbb{N}$ satisfies $s < \frac{\pi m}{2}$ and if T fulfills the oversampling condition $T > \frac{\pi m}{\pi m - 2s}\, L$, the uniform error $\|f - h\|_\infty$ is small too by Theorems 5.1 and 5.2. Note that we can choose $s \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{1\}$ with $s < \frac{\pi m}{2}$ independently.

5.2 Regularization with Sinh-Type Window Function

Let $m \in \mathbb{N} \setminus \{1\}$. Using the sinh-type window function (4.3), now we consider the regularized Shannon sampling formula for $\mathscr{F}_A$ with sampling density $T \ge \frac{m}{m-1}\, L$. The sinh-*type regularized Shannon sampling formula for* $\mathscr{F}_A$ with sampling density T takes the form

$$R^A_{\sinh,m} f(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\Big(\frac{n}{T}\Big)\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n)\, \varphi_{\sinh}\Big(t - \frac{n}{T}\Big), \quad t \in \mathbb{R}. \tag{5.5}$$

We will demonstrate that the uniform approximation error $\|f - R^A_{\sinh,m} f\|_\infty$ decays exponentially with respect to m.

Theorem 5.3 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. Let $m \in \mathbb{N}\setminus\{1\}$ and $T \ge \frac{m}{m-1} L$ be given. Let $\varphi_{\sinh} \in \Phi_{m/T}$ be the* sinh-*type window function* (4.3).

Then it holds

$$\|f - R^A_{\sinh,m} f\|_\infty \le 5\,\sqrt{L}\, \mathrm{e}^{-m\pi\,(T-L)/T}\, \|f\|_2. \tag{5.6}$$

Proof Note that $\beta = \frac{m\pi\,(T-L)}{T} \ge \pi$ by the assumptions $m \in \mathbb{N}\setminus\{1\}$ and $T \ge \frac{m}{m-1}\, L$. According to Theorem 4.3 we have

$$\|f - R^A_{\sinh,m} f\|_\infty \le \sqrt{L}\, \|f\|_2 \max_{\omega\in[-\pi L, \pi L]} \big|\Delta_{\sinh}(\omega)\big|$$

with

$$\Delta_{\sinh}(\omega) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi T}^{\omega+\pi T} \mathscr{F}\varphi_{\sinh}(\tau)\, \mathrm{d}\tau, \quad \omega \in [-\pi L, \pi L].$$

Following [18, p. 38, 7.58] we have

$$\mathscr{F}\varphi_{\sinh}(\tau) = \frac{m\sqrt{\pi}}{\sqrt{2}\,T\, \sinh\beta} \cdot \begin{cases} (1-\nu^2)^{-1/2}\, I_1(\beta\sqrt{1-\nu^2}) & |\nu| < 1, \\ (\nu^2-1)^{-1/2}\, J_1(\beta\sqrt{\nu^2-1}) & |\nu| > 1 \end{cases} \tag{5.7}$$

with the scaled frequency $\nu = \frac{m}{\beta T}\, \tau$. With this change of variable the function $\Delta_{\sinh}$ now reads as

$$\Delta_{\sinh}(\omega) = 1 - \frac{\beta T}{\sqrt{2\pi}\, m} \int_{-\nu_1(-\omega)}^{\nu_1(\omega)} \mathscr{F}\varphi_{\sinh}\Big(\frac{\beta T}{m}\, \nu\Big)\, \mathrm{d}\nu$$

with the linear increasing function $\nu_1(\omega) = \frac{m}{\beta T}(\omega + \pi T)$ for $\omega \in [-\pi L, \pi L]$. Since $\beta = \frac{m\pi\,(T-L)}{T}$, we have $\nu_1(-\pi L) = 1$ and $\nu_1(\omega) \ge 1$ for all $\omega \in [-\pi L, \pi L]$.

Now let $\Delta_{\mathrm{sinh}}(\omega) = \Delta_{\mathrm{sinh},1}(\omega) - \Delta_{\mathrm{sinh},2}(\omega)$ with

$$\begin{aligned}
\Delta_{\mathrm{sinh},1}(\omega) &= 1 - \frac{\beta T}{\sqrt{2\pi}\, m} \int_{-1}^{1} \mathscr{F}\varphi_{\mathrm{sinh}}\Big(\frac{\beta T}{m}\nu\Big)\,\mathrm{d}\nu,\\
\Delta_{\mathrm{sinh},2}(\omega) &= \frac{\beta T}{\sqrt{2\pi}\, m}\Big(\int_{-\nu_1(-\omega)}^{-1} + \int_1^{\nu_1(\omega)}\Big)\mathscr{F}\varphi_{\mathrm{sinh}}\Big(\frac{\beta T}{m}\nu\Big)\,\mathrm{d}\nu\\
&= \frac{\beta T}{\sqrt{2\pi}\, m}\Big(\int_{1}^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)}\Big)\mathscr{F}\varphi_{\mathrm{sinh}}\Big(\frac{\beta T}{m}\nu\Big)\,\mathrm{d}\nu.
\end{aligned}$$

In view of (5.7) these functions take the form

$$\begin{aligned}
\Delta_{\mathrm{sinh},1}(\omega) &= 1 - \frac{\beta}{2\sinh\beta}\int_{-1}^{1} \frac{I_1(\beta\sqrt{1-\nu^2})}{\sqrt{1-\nu^2}}\,\mathrm{d}\nu,\\
\Delta_{\mathrm{sinh},2}(\omega) &= \frac{\beta}{2\sinh\beta}\Big(\int_1^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)}\Big)\frac{J_1(\beta\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu.
\end{aligned}$$

Using [13, 6.681–3] and [2, 10.2.13], we get

$$\int_{-1}^{1} \frac{I_1(\beta\sqrt{1-\nu^2})}{\sqrt{1-\nu^2}}\,\mathrm{d}\nu = \int_{-\pi/2}^{\pi/2} I_1(\beta\cos\sigma)\,\mathrm{d}\sigma = \pi\,\Big(I_{1/2}\Big(\frac{\beta}{2}\Big)\Big)^2 = \frac{4}{\beta}\,\Big(\sinh\frac{\beta}{2}\Big)^2$$

and hence

$$\Delta_{\mathrm{sinh},1}(\omega) = 1 - \frac{2\Big(\sinh\frac{\beta}{2}\Big)^2}{\sinh\beta} = \frac{2\,\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}}.$$

By [13, 6.645–1] we have

$$\int_1^{\infty} \frac{J_1(\beta\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu = I_{1/2}\Big(\frac{\beta}{2}\Big)\,K_{1/2}\Big(\frac{\beta}{2}\Big) = \frac{1-\mathrm{e}^{-\beta}}{\beta} > 0,$$

where $I_{1/2}$ and $K_{1/2}$ are modified Bessel functions of half order (see [2, 10.2.13, 10.2.14, and 10.2.17]). Numerical experiments have shown that

$$\Big|\int_1^{W} \frac{J_1\Big(\beta\,\sqrt{\nu^2-1}\Big)}{\sqrt{\nu^2-1}}\,\mathrm{d}\nu\Big| \le \frac{3\,\Big(1-\mathrm{e}^{-\beta}\Big)}{2\,\beta}$$

for all $W > 1$ and $\beta \ge \pi$. Thus we obtain

$$\big|\Delta_{\mathrm{sinh},2}(\omega)\big| \le \frac{\beta}{2\,\sinh\beta}\,\frac{3\,\Big(1-\mathrm{e}^{-\beta}\Big)}{\beta} = \frac{3\,\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}}.$$

Altogether the estimate

$$\left|\Delta_{\sinh}(\omega)\right| \le \left|\Delta_{\sinh,1}(\omega) - \Delta_{\sinh,2}(\omega)\right| \le \frac{5\,\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}} < 5\,\mathrm{e}^{-\beta}$$

holds for all $\omega \in [-\pi L,\, \pi L]$. This implies

$$E(m, L, T) \le 5\,\sqrt{L}\,\mathrm{e}^{-m\pi\,(T-L)/T},$$

which completes the proof. □

Now we show that for the sinh-type regularized Shannon sampling formula (5.5) with respect to $\mathscr{F}_A$, the uniform perturbation error only grows as $\mathcal{O}(m)$, if $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ and if T fulfills the oversampling condition $T \ge \frac{m}{m-1}\,L$.

Theorem 5.4 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with some $L > 0$. Assume that $m \in \mathbb{N}\setminus\{1\}$ and $T \ge \frac{m}{m-1} L$ are given. Let $\varphi_{\sinh} \in \Phi_{m/T}$ be the* sinh*-type window function (4.3). Furthermore, let $\tilde f\left(\frac{n}{T}\right) = f\left(\frac{n}{T}\right) + \varepsilon_n$, $n \in \mathbb{Z}$, be noisy samples with complex error terms ε_n which are uniformly bounded by $|\varepsilon_n| \le \varepsilon$ with $0 < \varepsilon \ll 1$.*

Then the sinh*-type regularized Shannon sampling formula (5.5) for $\mathscr{F}_A$ with sampling density T is numerically robust and it holds*

$$\|R^A_{\sinh,m} f - h\|_\infty \le \varepsilon \left(2 + \frac{4m}{T}\right),$$

where h is given by

$$h(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{T}\right) \rho_A\left(\frac{n}{T}\right) \operatorname{sinc}(Tt - n)\, \varphi_{\sinh}\left(t - \frac{n}{T}\right), \quad t \in \mathbb{R}.$$

Proof By Theorem 4.5 we only have to determine the value $\mathscr{F}\varphi_{\sinh}(0)$ for the sinh-type window function (4.3). From (5.7) it follows that

$$\mathscr{F}\varphi_{\sinh}(0) = \frac{m\,\sqrt{\pi}}{\sqrt{2}\,T\,\sinh(\beta)}\,I_1(\beta).$$

Applying the inequality $\sqrt{2\pi\beta}\,\mathrm{e}^{-\beta}\,I_1(\beta) < 1$ for $\beta > 0$ (see [20, Lemma 7]), we find that

$$\mathscr{F}\varphi_{\sinh}(0) < \frac{m\,\mathrm{e}^{\beta}}{2\,\sqrt{\beta}\,T\,\sinh(\beta)} = \frac{\sqrt{m}}{\sqrt{T\,(T-L)}\,\left(1-\mathrm{e}^{-2\beta}\right)}.$$

Under the oversampling condition $T \geq \frac{m}{m-1} L$ it holds

$$\beta = \pi m \, \frac{T - L}{T} \geq \pi$$

for all $m \in \mathbb{N} \setminus \{1\}$. Hence we obtain

$$\frac{1}{\sqrt{T - L}} \leq \frac{\sqrt{m}}{\sqrt{T}}.$$

Therefore by Theorem 4.5 we can estimate

$$\begin{aligned} \|h - R_{\sinh,m}^{A} f\|_{\infty} &\leq \varepsilon \left(2 + \sqrt{2\pi}\, \mathscr{F} \varphi_{\sinh}(0)\right) \\ &\leq \varepsilon \left(2 + \frac{2\,\sqrt{\pi}}{1 - \mathrm{e}^{-2\beta}}\, \frac{m}{T}\right). \end{aligned}$$

Then by $\beta \geq \pi$ it follows

$$\frac{2\,\sqrt{\pi}}{1 - \mathrm{e}^{-2\beta}} \leq \frac{2\,\sqrt{\pi}}{1 - \mathrm{e}^{-2\pi}} = 3.551540\ldots < 4.$$

This completes the proof. □

5.3 Regularization with Continuous Kaiser–Bessel Window Function

Let $m \in \mathbb{N}\backslash\{1\}$ and $T \geq \frac{m}{m-1} L$ be given. Using the continuous Kaiser–Bessel window function (4.4), we finally consider the regularized Shannon sampling formula for $\mathscr{F}_A$. Then the *continuous Kaiser–Bessel regularized Shannon sampling formula* for $\mathscr{F}_A$ with sampling density T takes the form

$$R_{\mathrm{cKB},m}^{A} f(t) = \overline{\rho_A(t)} \sum_{n \in \mathbb{Z}} f\Big(\frac{n}{T}\Big)\, \rho_A\Big(\frac{n}{T}\Big)\, \mathrm{sinc}(Tt - n)\, \varphi_{\mathrm{cKB}}\Big(t - \frac{n}{T}\Big), \quad t \in \mathbb{R}. \tag{5.8}$$

Now we show that the uniform approximation error $\|f - R_{\mathrm{cKB},m}^{A} f\|_{\infty}$ decays exponentially with respect to m.

Theorem 5.5 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. Let $m \in \mathbb{N}\backslash\{1\}$ and $T \geq \frac{m}{m-1} L$ be given. Further let $\varphi_{\mathrm{cKB}} \in \Phi_{m/T}$ be the continuous Kaiser–Bessel window function* (4.4).

Then it holds

$$\|f - R_{\mathrm{cKB},m}^{A} f\|_{\infty} \leq \frac{\sqrt{L}}{I_0(\beta) - 1} \left(\frac{1}{2} + 4m\, \frac{T - L}{T}\right) \|f\|_2. \tag{5.9}$$

Proof Note that $\beta = \frac{m\pi\,(T-L)}{T} \geq \pi$ by our assumptions $m \in \mathbb{N}\backslash\{1\}$ and $T \geq \frac{m}{m-1}\,L$. From Theorem 4.3 it follows that

$$\|f - R_{\mathrm{cKB},m}^{A} f\|_\infty \leq \sqrt{L}\,\|f\|_2 \max_{\omega\in[-\pi L,\pi L]} \big|\Delta_{\mathrm{cKB}}(\omega)\big|$$

with

$$\Delta_{\mathrm{cKB}}(\omega) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\omega-\pi T}^{\omega+\pi T} \mathscr{F}\varphi_{\mathrm{cKB}}(\tau)\,\mathrm{d}\tau, \quad \omega \in [-\pi L, \pi L].$$

According to [18, p. 3, 1.1, and p. 95, 18.31], the Fourier transform of (4.4) has the form

$$\mathscr{F}\varphi_{\mathrm{cKB}}(\tau) = \frac{\sqrt{2}\,m}{\sqrt{\pi}\,T\,(I_0(\beta)-1)} \cdot \begin{cases} \Big(\frac{\sinh(\beta\sqrt{1-\nu^2})}{\beta\sqrt{1-\nu^2}} - \frac{\sin(\beta\nu)}{\beta\nu}\Big) & 0 < |\nu| < 1, \\ \Big(\frac{\sin(\beta\sqrt{\nu^2-1})}{\beta\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\Big) & |\nu| > 1, \end{cases} \tag{5.10}$$

with the scaled frequency $\nu = \frac{m}{\beta T}\,\tau$. Using this substitution, the function Δ_{cKB} now reads as

$$\Delta_{\mathrm{cKB}}(\omega) = 1 - \frac{\beta T}{\sqrt{2\pi}\,m} \int_{-\nu_1(-\omega)}^{\nu_1(\omega)} \mathscr{F}\varphi_{\mathrm{cKB}}\Big(\frac{\beta T}{m}\,\nu\Big)\,\mathrm{d}\nu$$

with the linear increasing function $\nu_1(\omega) = \frac{m}{\beta T}\,(\omega + \pi T)$ for $\omega \in [-\pi L, \pi L]$. Since $\beta = \frac{m\pi\,(T-L)}{T}$, we have $\nu_1(-\pi L) = 1$ and $\nu_1(\omega) \geq 1$ for all $\omega \in [-\pi L, \pi L]$.

Now let $\Delta_{\mathrm{cKB}}(\omega) = \Delta_{\mathrm{cKB},1}(\omega) - \Delta_{\mathrm{cKB},2}(\omega)$ with

$$\begin{aligned} \Delta_{\mathrm{cKB},1}(\omega) &= 1 - \frac{\beta T}{\sqrt{2\pi}\,m} \int_{-1}^{1} \mathscr{F}\varphi_{\mathrm{cKB}}\Big(\frac{\beta T}{m}\,\nu\Big)\,\mathrm{d}\nu, \\ \Delta_{\mathrm{cKB},2}(\omega) &= \frac{\beta T}{\sqrt{2\pi}\,m} \Big(\int_{-\nu_1(-\omega)}^{-1} + \int_{1}^{\nu_1(\omega)}\Big) \mathscr{F}\varphi_{\mathrm{cKB}}\Big(\frac{\beta T}{m}\,\nu\Big)\,\mathrm{d}\nu \\ &= \frac{\beta T}{\sqrt{2\pi}\,m} \Big(\int_{1}^{\nu_1(-\omega)} + \int_{1}^{\nu_1(\omega)}\Big) \mathscr{F}\varphi_{\mathrm{cKB}}\Big(\frac{\beta T}{m}\,\nu\Big)\,\mathrm{d}\nu. \end{aligned}$$

Using (5.10), these functions take the form

$$\begin{aligned} \Delta_{\mathrm{cKB},1}(\omega) &= 1 - \frac{\beta}{\pi\,(I_0(\beta)-1)} \int_{-1}^{1} \Big(\frac{\sinh(\beta\sqrt{1-\nu^2})}{\beta\sqrt{1-\nu^2}} - \frac{\sin(\beta\nu)}{\beta\nu}\Big)\,\mathrm{d}\nu, \\ \Delta_{\mathrm{cKB},2}(\omega) &= \frac{\beta}{\pi\,(I_0(\beta)-1)} \Big(\int_{1}^{\nu_1(-\omega)} + \int_{1}^{\nu_1(\omega)}\Big)\Big(\frac{\sin(\beta\sqrt{\nu^2-1})}{\beta\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\Big)\,\mathrm{d}\nu. \end{aligned}$$

By [13, 3.997–1] we have

$$\int_{-1}^{1} \frac{\sinh\left(\beta\sqrt{1-\nu^2}\right)}{\beta\sqrt{1-\nu^2}}\,\mathrm{d}\nu = \frac{2}{\beta}\int_{0}^{1} \frac{\sinh\left(\beta\sqrt{1-\nu^2}\right)}{\sqrt{1-\nu^2}}\,\mathrm{d}\nu$$
$$= \frac{2}{\beta}\int_{0}^{\pi/2} \sinh(\beta\cos\sigma)\,\mathrm{d}\sigma = \frac{\pi}{\beta}\,\mathbf{L}_0(\beta),$$

where $\mathbf{L}_0$ denotes the *modified Struve function* (see [2, 12.2.1]) given by

$$\mathbf{L}_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{\left(\Gamma\left(k+\frac{3}{2}\right)\right)^2} = \frac{2x}{\pi}\sum_{k=0}^{\infty} \frac{x^{2k}}{\left((2k+1)!!\right)^2}, \quad x \in \mathbb{R}.$$

Note that the function $I_0(x) - \mathbf{L}_0(x)$ is completely monotonic on $[0, \infty)$ (see [4, Theorem 1]) and tends to zero as $x \to \infty$. Applying the *sine integral function*

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

implies

$$\int_{-1}^{1} \frac{\sin(\beta\nu)}{\beta\nu}\,\mathrm{d}\nu = 2\int_{0}^{1} \frac{\sin(\beta\nu)}{\beta\nu}\,\mathrm{d}\nu = \frac{2}{\beta}\,\mathrm{Si}(\beta).$$

Hence we obtain

$$\Delta_{\mathrm{cKB},1}(\omega) = 1 - \frac{1}{I_0(\beta)-1}\left(\mathbf{L}_0(\beta) - \frac{2}{\pi}\,\mathrm{Si}(\beta)\right)$$
$$= \frac{1}{I_0(\beta)-1}\left(I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\right).$$

For $\beta = \frac{m\pi\,(T-L)}{T} \geq \pi$ it holds by [16, Lemma 5.1] that

$$\left|I_0(\beta) - \mathbf{L}_0(\beta) - 1 + \frac{2}{\pi}\,\mathrm{Si}(\beta)\right| < \frac{1}{2}.$$

Further it is known that $I_0(\beta) > 1$ for $\beta > 0$, which implies $\left|\Delta_{\mathrm{cKB},1}(\omega)\right| < \frac{1}{2\left(I_0(\beta)-1\right)}$ for $\omega \in [-\pi L, \pi L]$.

Now we estimate $\left|\Delta_{\mathrm{cKB},2}(\omega)\right|$ for $\omega \in [-\pi L, \pi L]$ by the triangle inequality as follows

$$\left|\Delta_{\mathrm{cKB},2}(\omega)\right| \leq \frac{\beta}{\pi\left(I_0(\beta)-1\right)}\left(\int_1^{\nu_1(-\omega)} + \int_1^{\nu_1(\omega)}\right)\left|\frac{\sin\left(\beta\sqrt{\nu^2-1}\right)}{\beta\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\,\nu}\right|\mathrm{d}\nu.$$

By [20, Lemma 4], we have for $\nu \ge 1$ that

$$\Big|\frac{\sin\big(\beta\,\sqrt{\nu^2-1}\big)}{\beta\,\sqrt{\nu^2-1}} - \frac{\sin(\beta\nu)}{\beta\nu}\Big| \le \frac{2}{\nu^2}.$$

Thus we conclude that

$$\big|\Delta_{\mathrm{cKB},2}(\omega)\big| \le \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)} \int_1^\infty \frac{1}{\nu^2}\,\mathrm{d}\nu = \frac{4\beta}{\pi\,\big(I_0(\beta)-1\big)}.$$

Therefore we obtain for $\omega \in [-\pi L, \pi L]$ that

$$\begin{aligned}\big|\Delta_{\mathrm{cKB}}(\omega)\big| &\le \big|\Delta_{\mathrm{cKB},1}(\omega)\big| + \big|\Delta_{\mathrm{cKB},2}(\omega)\big| \le \frac{1}{I_0(\beta)-1}\Big(\frac{1}{2}+\frac{4\beta}{\pi}\Big)\\ &= \frac{1}{I_0(\beta)-1}\Big(\frac{1}{2}+4m\,\frac{T-L}{T}\Big).\end{aligned}$$

This completes the proof. □

Remark 5.6 By [3, 16, Lemma 5.2], the function $\mathrm{e}^x/\big(x\,I_0(x)-x\big)$ is strictly decreasing on $\big[\pi,\ \infty\big)$. Under the oversampling condition $T \ge \frac{m}{m-1}\,L$ we have

$$\beta = \pi m\,\frac{T-L}{T} \ge \pi \tag{5.11}$$

for all $m \in \mathbb{N}\setminus\{1\}$. Hence one can estimate

$$0 < \frac{\mathrm{e}^\beta}{\beta\ I_0(\beta)-\beta} \le \frac{2\,\mathrm{e}^\pi}{\pi\ \ I_0(\pi)-\pi} = 1.6444967\ldots < \frac{7}{4}. \tag{5.12}$$

Thus we conclude that

$$0 < \frac{1}{I_0(\beta)-1}\Big(\frac{1}{2}+\frac{4\beta}{\pi}\Big) < \Big(\frac{7\beta}{8}+\frac{7\beta^2}{\pi}\Big)\,\mathrm{e}^{-\beta}.$$

Then by Theorem 5.5, the approximation error of the continuous Kaiser–Bessel regularized Shannon sampling formula (5.8) decreases exponentially with respect to m, if T fulfills the oversampling condition $T \ge \frac{m}{m-1}\,L$.

Finally we show that for the continuous Kaiser–Bessel regularized Shannon sampling formula (5.8) related to $\mathscr{F}_A$, the uniform perturbation error only grows as $\mathcal{O}(m)$, if $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ and T fulfills the oversampling condition $T \ge \frac{m}{m-1}\,L$.

Theorem 5.7 *Let $f \in L^2(\mathbb{R})$ be $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. Let $m \in \mathbb{N}\setminus\{1\}$ and $T \ge \frac{m}{m-1}\,L$ be given. Further let $\varphi_{\mathrm{cKB}} \in \Phi_{m/T}$ be the continuous*

Kaiser–Bessel window function (4.4)*. Furthermore, let* $\tilde{f}\big(\frac{n}{T}\big) = f\big(\frac{n}{T}\big) + \varepsilon_n$*,* $n \in \mathbb{Z}$*, be noisy samples with complex error terms* ε_n *which are uniformly bounded by* $|\varepsilon_n| \le \varepsilon$ *with* $0 < \varepsilon \ll 1$.

Then the continuous Kaiser–Bessel regularized Shannon sampling formula (5.8) *for* $\mathscr{F}_A$ *with sampling density* T *is numerically robust and it holds*

$$\|R^A_{\mathrm{cKB},m} f - h\|_\infty \le \varepsilon\,\Big(2 + \frac{7m}{4T}\Big),$$

where h *is defined by*

$$h(t) = \overline{\rho_A(t)} \sum_{n\in\mathbb{Z}} \tilde{f}\Big(\frac{n}{T}\Big)\,\rho_A\Big(\frac{n}{T}\Big)\,\mathrm{sinc}(Tt - n)\,\varphi_{\mathrm{cKB}}\Big(t - \frac{n}{T}\Big), \quad t \in \mathbb{R}.$$

Proof By Theorem 4.5 we only have to estimate the value $\mathscr{F}\varphi_{\mathrm{cKB}}(0)$ for the continuous Kaiser–Bessel window function (4.4). From (5.10) it follows that

$$\mathscr{F}\varphi_{\mathrm{cKB}}(0) = \frac{\sqrt{2}\,m}{\sqrt{\pi}\,T\,\big(I_0(\beta)-1\big)}\Big(\frac{\sinh\beta}{\beta} - 1\Big) \tag{5.13}$$

$$= \frac{m\,\mathrm{e}^{\beta}}{\sqrt{2\pi}\,\beta\,T\,\big(I_0(\beta)-1\big)}\big(1 - \mathrm{e}^{-2\beta} - 2\beta\,\mathrm{e}^{-\beta}\big) \tag{5.14}$$

$$< \frac{m\,\mathrm{e}^{\beta}}{\sqrt{2\pi}\,\beta\,T\,\big(I_0(\beta)-1\big)}. \tag{5.15}$$

By the oversampling condition $T \ge \frac{m}{m-1}\,L$ we have the estimate (5.11). Thus by (5.12) it holds

$$0 < \frac{\mathrm{e}^{\beta}}{\beta\,I_0(\beta) - \beta} < \frac{7}{4}$$

and hence

$$\sqrt{2\pi}\,\mathscr{F}\varphi_{\mathrm{cKB}}(0) < \frac{7m}{4T}.$$

This completes the proof. □

Using the special window functions (4.2), (4.3), and (4.4), we compare the uniform approximation error of the regularized Shannon sampling formulas for $\mathscr{F}_A$. Assume that $f \in L^2(\mathbb{R})$ is $\mathscr{F}_A$-bandlimited in $[-\pi L\,|b|, \pi L\,|b|]$ with $L > 0$. We consider only the oversampling case $T = 2L$.

For the B-spline window function $\varphi = \varphi_{\mathrm{B}} \in \Phi_{m/T}$ of order $2\,s$ with $s = \lfloor m/2 \rfloor$, the uniform approximation error can be estimated by (5.2) in the form

$$\|f - R^A_{\varphi,m} f\|_\infty \le \Big(\frac{4s}{m\pi}\Big)^{2s-1}\sqrt{L}\,\|f\|_2, \quad m \in \mathbb{N}\setminus\{1\}.$$

Table 2 Upper bounds of the uniform approximation error $\|f - R_{\varphi,m}^A f\|_\infty$ for special window functions $\varphi \in \Phi_{m/T}$ in the case $T = 2L$

Window function $\varphi \in \Phi_{m/T}$	Truncation parameter m	Upper bound of $\|f - R_{\varphi,m}^A f\|_\infty$
$\varphi = \varphi_{\mathrm{B}}$	3	$4.24 \times 10^{-1} \sqrt{L}\, \|f\|_2$
	4	$2.58 \times 10^{-1} \sqrt{L}\, \|f\|_2$
	5	$1.32 \times 10^{-1} \sqrt{L}\, \|f\|_2$
	6	$1.04 \times 10^{-1} \sqrt{L}\, \|f\|_2$
$\varphi = \varphi_{\mathrm{sinh}}$	3	$4.49 \times 10^{-2} \sqrt{L}\, \|f\|_2$
	4	$9.33 \times 10^{-3} \sqrt{L}\, \|f\|_2$
	5	$1.94 \times 10^{-3} \sqrt{L}\, \|f\|_2$
	6	$4.03 \times 10^{-4} \sqrt{L}\, \|f\|_2$
$\varphi = \varphi_{\mathrm{cKB}}$	3	$3.23 \times 10^{-1} \sqrt{L}\, \|f\|_2$
	4	$9.87 \times 10^{-2} \sqrt{L}\, \|f\|_2$
	5	$2.82 \times 10^{-2} \sqrt{L}\, \|f\|_2$
	6	$7.65 \times 10^{-3} \sqrt{L}\, \|f\|_2$

For the sinh-type window function $\varphi = \varphi_{\mathrm{sinh}} \in \Phi_{m/T}$, the uniform approximation error can be estimated by (5.6) in the form

$$\|f - R_{\varphi,m}^A f\|_\infty \le 5\, \mathrm{e}^{-m\pi/2} \sqrt{L}\, \|f\|_2, \quad m \in \mathbb{N} \setminus \{1\}.$$

For the continuous Kaiser–Bessel window function $\varphi = \varphi_{\mathrm{cKB}} \in \Phi_{m/T}$, we obtain by (5.9) the estimate

$$\|f - R_{\varphi,m}^A f\|_\infty \le \frac{1}{I_0(m\pi/2) - 1} \left(\frac{1}{2} + 2m\right) \sqrt{L}\, \|f\|_2, \quad m \in \mathbb{N} \setminus \{1\}.$$

In Table 2, we see that the regularized Shannon sampling formulas for $\mathscr{F}_A$ with the sinh-type window function (4.3) or the continuous Kaiser–Bessel window function (4.4) are very accurate for $m \ge 4$. Further, these regularized Shannon sampling formulas for $\mathscr{F}_A$ can easily be computed, require less samples, and are numerically robust for noisy samples.

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