Z. Angew. Math. Phys. (2025) 76:234

https://doi.org/10.1007/s00033-025-02635-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

On the well-posedness of a certain model with the bi-Laplacian appearing in the mathematical biology

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Abstract. The work is devoted to the global well-posedness in $W^{(1,4),2}(\mathbb{R}\times\mathbb{R}^+)$ of the integro-differential problem involving the square of the one dimensional Laplace operator along with the drift term. Our proof is based on a fixed point technique. Moreover, we provide the assumption leading to the existence of the nontrivial solution for the problem under the consideration. Such equation is relevant to the cell population dynamics in the Mathematical Biology.

Mathematics Subject Classification. 35K25, 35K57, 35R09.

Keywords. Integro-differential equations, Well-posedness, Bi-Laplacian, Sobolev spaces.

1. Introduction

The present article is devoted to the global well-posedness of the nonlocal reaction-diffusion equation with the constants $a \geq 0$ and $b \in \mathbb{R}$, namely

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + b\frac{\partial u}{\partial x} + au + \int_{-\infty}^{\infty} G(x-y)F(u(y,t),y)dy, \quad x \in \mathbb{R} \tag{1.1}$$

relevant to the cell population dynamics. We assume that the initial condition for (1.1) is

$$u(x,0) = u_0(x) \in H^4(\mathbb{R}). \tag{1.2}$$

The similar problem on the real line involving the fractional Laplacian in the context of the anomalous diffusion was treated in [16]. Note that the existence of stationary solutions of the integro-differential equations with the bi-Laplacian and the biological applications of such models but without a transport term were discussed in [24]. The cases on the whole real line and on a finite interval with periodic boundary conditions involving the drift term and the square root of the one dimensional negative Laplacian were covered in [15]. The work [14] deals with situation of the normal diffusion and the transport. Solvability of certain integro-differential equations with anomalous diffusion, transport and the cell influx/efflux was considered in [25]. Spatial structures and generalized traveling waves for an integro-differential equation were treated in [2]. Spatial patterns appearing in higher order models in physics and mechanics were covered in [22]. The article [23] is devoted to the emergence and propagation of patterns in nonlocal reaction- diffusion equations arising in the theory of speciation and containing the drift term. Pattern and waves for a model in population dynamics with nonlocal consumptions of resources were studied in [18]. The existence of steady states and travelling waves for the nonlocal Fisher-KPP equation was covered in [3]. In [4], the authors estimated the speed of propagation for KPP type problems in the periodic framework. Important applications to the theory of reaction-diffusion equations with non-Fredholm operators were developed in [7,8]. Fredholm structures, topological invariants and applications were covered in [9]. Evolution equations arising in the modeling of life sciences were considered in [11]. In work [17],

Published online: 29 October 2025

the reaction-diffusion systems with the nonlinearity depending on u and ∇u were systematically studied in unbounded domains. The large time behavior of solutions of fourth order parabolic equations and ϵ-entropy of their attractors were discussed in [13]. Lower estimate of the attractor dimension for a chemotaxis growth system was performed in [1]. The theory of finite and infinite dimensional attractors for evolution equations of mathematical physics was developed in [10]. Attractors for degenerate parabolic type equations were investigated in [12]. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations was established in [6]. Local and global existence for nonlocal multispecies advection–diffusion models were covered in [19].

The space variable x in our article is correspondent to the cell genotype, $u(x,t)$ denotes the cell density as a function of the genotype and time. The right side of (1.1) describes the evolution of the cell density via the cell proliferation, mutations and transport. The diffusion term corresponds to the change of genotype due to the small random mutations, and the integral term describes large mutations. The function $F(u,x)$ stands for the rate of cell birth depending on u and x (density dependent proliferation), and the kernel $G(x-y)$ gives the proportion of newly born cells which change their genotype from y to x. Let us assume that it depends on the distance between the genotypes.

The standard Fourier transform in our context equals to

$$\widehat{\phi}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-ipx} dx, \quad p \in \mathbb{R}. \tag{1.3}$$

Clearly, the estimate from above

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|\phi(x)\|_{L^1(\mathbb{R})} \tag{1.4}$$

is valid (see, e.g., [21]). Evidently, (1.4) implies that

$$\|p^4 \widehat{\phi}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \left\| \frac{d^4\phi}{dx^4} \right\|_{L^1(\mathbb{R})}. \tag{1.5}$$

Let us suppose that the conditions below on the integral kernel contained in equation (1.1) are satisfied.

Assumption 1.1. Let $G(x) : \mathbb{R} \to \mathbb{R}$ be nontrivial, such that $G(x),\ \dfrac{d^4G(x)}{dx^4} \in L^1(\mathbb{R})$.

This allows us to define the auxiliary expression

$$g := \sqrt{\|G(x)\|^2_{L^1(\mathbb{R})} + \left\| \frac{d^4G(x)}{dx^4} \right\|^2_{L^1(\mathbb{R})}}. \tag{1.6}$$

Thus, $0 < g < \infty$.

From the perspective of the applications, the space dimension is not limited to $d = 1$ since the space variable corresponds to the cell genotype but not to the usual physical space. We have the Sobolev space

$$H^4(\mathbb{R}) := \left\{ \phi(x) : \mathbb{R} \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}),\ \frac{d^4\phi}{dx^4} \in L^2(\mathbb{R}) \right\}. \tag{1.7}$$

It is equipped with the norm

$$\|\phi\|^2_{H^4(\mathbb{R})} := \|\phi\|^2_{L^2(\mathbb{R})} + \left\| \frac{d^4\phi}{dx^4} \right\|^2_{L^2(\mathbb{R})}. \tag{1.8}$$

For establishing the global well-posedness of problem (1.1), (1.2), we will use the function space

$$W^{(1,4),2}(\mathbb{R}\times[0,T]) := \left\{ u(x,t):\mathbb{R}\times[0,T]\to\mathbb{R} \;\middle|\; u(x,t),\ \frac{\partial^4 u}{\partial x^4},\ \frac{\partial u}{\partial t}\in L^2(\mathbb{R}\times[0,T]) \right\}, \tag{1.9}$$

so that

$$\|u(x,t)\|^2_{W^{(1,4),2}(\mathbb{R}\times[0,T])} := \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\mathbb{R}\times[0,T])} + \left\|\frac{\partial^4 u}{\partial x^4}\right\|^2_{L^2(\mathbb{R}\times[0,T])} + \|u\|^2_{L^2(\mathbb{R}\times[0,T])}, \tag{1.10}$$

where $T>0$. In definition (1.10), we have

$$\|u\|^2_{L^2(\mathbb{R}\times[0,T])} := \int_0^T\int_{-\infty}^{\infty}|u(x,t)|^2dxdt.$$

Throughout the work, we will also use the norm

$$\|u(x,t)\|^2_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty}|u(x,t)|^2dx.$$

Assumption 1.2. Function $F(u,x):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is satisfying the Caratheodory condition (see [20]), so that

$$|F(u,x)|\le k|u|+h(x) \quad for \quad u\in\mathbb{R}, \quad x\in\mathbb{R} \tag{1.11}$$

with a constant $k>0$ and $h(x):\mathbb{R}\to\mathbb{R}^+,\ h(x)\in L^2(\mathbb{R})$. Moreover, it is a Lipschitz continuous function, such that

$$|F(u_1,x)-F(u_2,x)|\le l|u_1-u_2| \quad for \quad any \quad u_{1,2}\in\mathbb{R}, \quad x\in\mathbb{R} \tag{1.12}$$

with a constant $l>0$.

In our article $\mathbb{R}^+$ stands for the nonnegative semi-axis. The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was covered in [5]. The nonlinear function involved there was allowed to have a sublinear growth.

We apply the standard Fourier transform (1.3) to both sides of problem (1.1), (1.2). This gives us

$$\frac{\partial\widehat{u}}{\partial t} = [-p^4+ibp+a]\widehat{u}+\sqrt{2\pi}\widehat{G}(p)\widehat{f}_u(p,t), \tag{1.13}$$

$$\widehat{u(x,0)}(p)=\widehat{u_0}(p). \tag{1.14}$$

In formula (1.13) and below $\widehat{f}_u(p,t)$ will denote the Fourier image of $F(u(x,t),x)$. Obviously,

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{u}(p,t)e^{ipx}dp, \quad \frac{\partial u}{\partial t}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial\widehat{u}(p,t)}{\partial t}e^{ipx}dp,$$

where $x\in\mathbb{R},\ t\ge 0$. By virtue of the Duhamel's principle, we can reformulate problem (1.13), (1.14) as

$$\widehat{u}(p,t) = e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p)+\int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)\widehat{f}_u(p,s)ds. \tag{1.15}$$

Related to (1.15), we have the auxiliary equation

$$\widehat{u}(p,t) = e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p) + \int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)\widehat{f_v}(p,s)ds. \tag{1.16}$$

Here $\widehat{f_v}(p,s)$ stands for the Fourier image of $F(v(x,s),x)$ under transform (1.3), where $v(x,t)$ is an arbitrary function belonging to $W^{(1,4),2}(\mathbb{R}\times[0,T])$.

We introduce the operator $\tau_{a,b}$, so that $u = \tau_{a,b}v$, where u solves (1.16). The main result of the article is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 be valid and*

$$gl\sqrt{T^2e^{2aT}(1+2[a+|b|+1]^2)+2} < 1 \tag{1.17}$$

with the constant g defined in (1.6) and the Lipschitz constant l introduced in (1.12). Then equation (1.16) defines the map $\tau_{a,b}: W^{(1,4),2}(\mathbb{R}\times[0,T]) \to W^{(1,4),2}(\mathbb{R}\times[0,T])$, which is a strict contraction. The unique fixed point $w(x,t)$ of this map $\tau_{a,b}$ is the only solution of problem (1.1), (1.2) in $W^{(1,4),2}(\mathbb{R}\times[0,T])$.

The final proposition of our work is devoted to the global well-posedness for our equation.

Corollary 1.4. *Let the assumptions of Theorem 1.3 hold. Then problem (1.1), (1.2) possesses a unique solution $w(x,t) \in W^{(1,4),2}(\mathbb{R}\times\mathbb{R}^+)$. Such solution is nontrivial for $x\in\mathbb{R}$ and $t\in\mathbb{R}^+$ provided the intersection of supports of the Fourier images of functions $supp\widehat{F(0,x)}\cap supp\widehat{G}$ is a set of nonzero Lebesgue measure on the real line.*

Let us proceed to the proof of our main result.

2. The well-posedness of the equation

Proof of Theorem 1.3. We choose arbitrarily $v(x,t)\in W^{(1,4),2}(\mathbb{R}\times[0,T])$. It is not difficult to see that the first term in the right side of (1.16) is contained in $L^2(\mathbb{R}\times[0,T])$. Evidently,

$$\|e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} e^{-2tp^4}e^{2at}|\widehat{u_0}(p)|^2dp \le e^{2at}\|u_0\|^2_{L^2(\mathbb{R})},$$

so that

$$\|e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p)\|^2_{L^2(\mathbb{R}\times[0,T])} = \int_0^T \|e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p)\|^2_{L^2(\mathbb{R})}dt \le \int_0^T e^{2at}\|u_0\|^2_{L^2(\mathbb{R})}dt.$$

The right hand side of the last inequality is equal to $\dfrac{e^{2aT}-1}{2a}\|u_0\|^2_{L^2(\mathbb{R})}$ if $a>0$ and $T\|u_0\|^2_{L^2(\mathbb{R})}$ for $a=0$. Thus,

$$e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p)\in L^2(\mathbb{R}\times[0,T]). \tag{2.1}$$

Clearly, the upper bound on the norm of the second term in the right side of (1.16)

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|_{L^2(\mathbb{R})} \leq \int_0^t \left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})} ds.$$

holds. We have

$$\left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} e^{-2(t-s)p^4} e^{2a(t-s)} 2\pi |\widehat{G}(p)|^2 |\widehat{f}_v(p,s)|^2 dp. \tag{2.2}$$

Let us use inequality (1.4) to derive the estimate from above on the right side of (2.2) as

$$e^{2a(t-s)} 2\pi \|\widehat{G}(p)\|_{L^\infty(\mathbb{R})}^2 \|F(v(x,s),x)\|_{L^2(\mathbb{R})}^2 \leq e^{2aT} \|G(x)\|_{L^1(\mathbb{R})}^2 \|F(v(x,s),x)\|_{L^2(\mathbb{R})}^2.$$

Hence,

$$\left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})} \leq e^{aT} \|G(x)\|_{L^1(\mathbb{R})} \|F(v(x,s),x)\|_{L^2(\mathbb{R})}.$$

By virtue of (1.11), we obtain

$$\|F(v(x,s),x)\|_{L^2(\mathbb{R})} \leq k \|v(x,s)\|_{L^2(\mathbb{R})} + \|h(x)\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Then

$$\left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})} \leq e^{aT} \|G(x)\|_{L^1(\mathbb{R})} \{k \|v(x,s)\|_{L^2(\mathbb{R})} + \|h(x)\|_{L^2(\mathbb{R})}\},$$

so that

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|_{L^2(\mathbb{R})} \leq k e^{aT} \|G(x)\|_{L^1(\mathbb{R})} \int_0^T \|v(x,s)\|_{L^2(\mathbb{R})} ds + T e^{aT} \|G(x)\|_{L^1(\mathbb{R})} \|h(x)\|_{L^2(\mathbb{R})}.$$

By means of the Schwarz inequality

$$\int_0^T \|v(x,s)\|_{L^2(\mathbb{R})} ds \leq \sqrt{\int_0^T \|v(x,s)\|_{L^2(\mathbb{R})}^2 ds} \sqrt{T}. \tag{2.4}$$

This yields

$$\Big\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,s)ds \Big\|^2_{L^2(\mathbb{R})}$$
$$\leq e^{2aT}\|G(x)\|^2_{L^1(\mathbb{R})}\{k\sqrt{T}\|v(x,s)\|_{L^2(\mathbb{R}\times[0,T])} + T\|h(x)\|_{L^2(\mathbb{R})}\}^2.$$

We derive the upper bound on the norm as

$$\Big\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,s)ds \Big\|^2_{L^2(\mathbb{R}\times[0,T])}$$
$$= \int_0^T \Big\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,s)ds \Big\|^2_{L^2(\mathbb{R})} dt$$
$$\leq e^{2aT}\|G(x)\|^2_{L^1(\mathbb{R})}\{k\|v(x,s)\|_{L^2(\mathbb{R}\times[0,T])} + \sqrt{T}\|h(x)\|_{L^2(\mathbb{R})}\}^2 T^2 < \infty$$

under the stated assumptions for $v(x,s) \in W^{(1,4),2}(\mathbb{R}\times[0,T])$. Therefore,

$$\int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,s)ds \in L^2(\mathbb{R}\times[0,T]). \tag{2.5}$$

By virtue of (2.1), (2.5) and (1.16), we have

$$\widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.6}$$

This means that

$$u(x,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.7}$$

Let us recall formula (1.16). Thus,

$$p^4\widehat{u}(p,t)$$
$$= e^{t\{-p^4+ibp+a\}}p^4\widehat{u_0}(p) + \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi}p^4\widehat{G}(p)\widehat{f}_v(p,s)ds. \tag{2.8}$$

Consider the first term in the right side of (2.8). Clearly,

$$\|e^{t\{-p^4+ibp+a\}}p^4\widehat{u_0}(p)\|^2_{L^2(\mathbb{R}\times[0,T])} = \int_0^T \int_{-\infty}^{\infty} e^{-2tp^4}e^{2at}|p^4\widehat{u_0}(p)|^2 dpdt$$
$$\leq \int_0^T \int_{-\infty}^{\infty} e^{2at}|p^4\widehat{u_0}(p)|^2 dpdt.$$

Obviously, this equals to $\dfrac{e^{2aT}-1}{2a}\Big\|\dfrac{d^4u_0}{dx^4}\Big\|^2_{L^2(\mathbb{R})}$ for $a>0$ and $T\Big\|\dfrac{d^4u_0}{dx^4}\Big\|^2_{L^2(\mathbb{R})}$ when $a=0$. Therefore,

$$e^{t\{-p^4+ibp+a\}}p^4\widehat{u_0}(p) \in L^2(\mathbb{R}\times[0,T]). \tag{2.9}$$

We proceed to analyzing the second term in the right side of (2.8). Evidently,

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|_{L^2(\mathbb{R})} \leq \int_0^t \left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})} ds.$$

Note that

$$\left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) \right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} e^{-2(t-s)p^4} e^{2a(t-s)} 2\pi |p^4 \widehat{G}(p)|^2 |\widehat{f}_v(p,s)|^2 dp. \tag{2.10}$$

The right side of (2.10) can be estimated from above via inequality (1.5) as

$$2\pi e^{2aT} \|p^4 \widehat{G}(p)\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} |\widehat{f}_v(p,s)|^2 dp \leq e^{2aT} \left\| \frac{d^4 G}{dx^4} \right\|^2_{L^1(\mathbb{R})} \|F(v(x,s),x)\|^2_{L^2(\mathbb{R})}.$$

By means of (2.3) , we have

$$\left\| e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) \right\|_{L^2(\mathbb{R})} \leq e^{aT} \left\| \frac{d^4 G}{dx^4} \right\|_{L^1(\mathbb{R})} \{k \|v(x,s)\|_{L^2(\mathbb{R})} + \|h(x)\|_{L^2(\mathbb{R})}\}.$$

Then

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|_{L^2(\mathbb{R})} \leq k e^{aT} \left\| \frac{d^4 G}{dx^4} \right\|_{L^1(\mathbb{R})} \int_0^T \|v(x,s)\|_{L^2(\mathbb{R})} ds + T e^{aT} \left\| \frac{d^4 G}{dx^4} \right\|_{L^1(\mathbb{R})} \|h(x)\|_{L^2(\mathbb{R})}.$$

Let us recall estimate (2.4), which implies

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|^2_{L^2(\mathbb{R})} \leq e^{2aT} \left\| \frac{d^4 G}{dx^4} \right\|^2_{L^1(\mathbb{R})} \{k \|v(x,s)\|_{L^2(\mathbb{R} \times [0,T])} \sqrt{T} + \|h(x)\|_{L^2(\mathbb{R})} T\}^2.$$

This means that

$$\left\| \int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) ds \right\|^2_{L^2(\mathbb{R}\times[0,T])}$$

$$\leq e^{2aT} \left\| \frac{d^4 G}{dx^4} \right\|^2_{L^1(\mathbb{R})} \{k\|v(x,s)\|_{L^2(\mathbb{R}\times[0,T])} + \|h(x)\|_{L^2(\mathbb{R})}\sqrt{T}\}^2 T^2 < \infty$$

under the given conditions with $v(x,s) \in W^{(1,4),2}(\mathbb{R}\times[0,T])$. Therefore,

$$\int_0^t e^{(t-s)\{-p^4+ibp+a\}} \sqrt{2\pi} p^4 \widehat{G}(p) \widehat{f}_v(p,s) ds \in L^2(\mathbb{R}\times[0,T]). \tag{2.11}$$

(2.9), (2.11) and (2.8) yield that

$$p^4 \widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]), \tag{2.12}$$

so that

$$\frac{\partial^4 u}{\partial x^4} \in L^2(\mathbb{R}\times[0,T]). \tag{2.13}$$

Using (1.16), we easily derive

$$\frac{\partial \widehat{u}}{\partial t} = \{-p^4 + ibp + a\}\widehat{u}(p,t) + \sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,t). \tag{2.14}$$

According to (2.6),

$$a\widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.15}$$

We obtain the upper bound for the norm as

$$\|ibp\widehat{u}(p,t)\|^2_{L^2(\mathbb{R}\times[0,T])} = b^2 \int_0^T \left\{ \int_{|p|\leq 1} p^2 |\widehat{u}(p,t)|^2 dp + \int_{|p|>1} p^2 |\widehat{u}(p,t)|^2 dp \right\} dt$$

$$\leq b^2 \{\|\widehat{u}(p,t)\|^2_{L^2(\mathbb{R}\times[0,T])} + \|p^4\widehat{u}(p,t)\|^2_{L^2(\mathbb{R}\times[0,T])}\} < \infty$$

due to (2.6) and (2.12). Hence,

$$ibp\widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.16}$$

And

$$-p^4\widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]) \tag{2.17}$$

via (2.12). Let us combine statements (2.15), (2.16) and (2.17), which yields

$$(-p^4 + ibp + a)\widehat{u}(p,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.18}$$

We consider the remaining term in the right side of (2.14). Recall inequalities (1.4) and (2.3). Clearly,

$$\|\sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,t)\|^2_{L^2(\mathbb{R}\times[0,T])} \leq 2\pi\|\widehat{G}(p)\|^2_{L^\infty(\mathbb{R})} \int_0^T \|F(v(x,t),x)\|^2_{L^2(\mathbb{R})} dt$$

$$\leq \|G(x)\|^2_{L^1(\mathbb{R})} \int_0^T (k\|v(x,t)\|_{L^2(\mathbb{R})} + \|h(x)\|_{L^2(\mathbb{R})})^2 dt$$

$$\leq \|G(x)\|^2_{L^1(\mathbb{R})} \{2k^2\|v(x,t)\|^2_{L^2(\mathbb{R}\times[0,T])} + 2\|h(x)\|^2_{L^2(\mathbb{R})}T\} < \infty$$

under the stated assumptions for $v(x,t) \in W^{(1,4),2}(\mathbb{R}\times[0,T])$. Hence,

$$\sqrt{2\pi}\widehat{G}(p)\widehat{f}_v(p,t) \in L^2(\mathbb{R}\times[0,T]). \tag{2.19}$$

By means of equation (2.14) along with statements (2.18) and (2.19), we have

$$\frac{\partial \widehat{u}}{\partial t} \in L^2(\mathbb{R} \times [0, T]),$$

so that

$$\frac{\partial u}{\partial t} \in L^2(\mathbb{R} \times [0, T]). \tag{2.20}$$

Using the definition of the norm (1.10) along with (2.7), (2.13) and (2.20), we derive that for the function uniquely determined by (1.16),

$$u(x, t) \in W^{(1,4),2}(\mathbb{R} \times [0, T]).$$

This means that under the given conditions equation (1.16) defines a map

$$\tau_{a,b} : W^{(1,4),2}(\mathbb{R} \times [0, T]) \to W^{(1,4),2}(\mathbb{R} \times [0, T]).$$

Let us establish that under the stated assumptions such map is a strict contraction. We choose arbitrarily $v_{1,2}(x, t) \in W^{(1,4),2}(\mathbb{R} \times [0, T])$. By virtue of the argument above, $u_{1,2} := \tau_{a,b} v_{1,2} \in W^{(1,4),2}(\mathbb{R} \times [0, T])$. According to formula (1.16), we have

$$\widehat{u_1}(p, t) = e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p) + \int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)\widehat{f}_{v_1}(p, s)ds, \tag{2.21}$$

$$\widehat{u_2}(p, t) = e^{t\{-p^4+ibp+a\}}\widehat{u_0}(p) + \int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)\widehat{f}_{v_2}(p, s)ds, \tag{2.22}$$

where $\widehat{f}_{v_j}(p, s)$ with $j = 1, 2$ denotes the Fourier image of $F(v_j(x, s), x)$ under transform (1.3). From system (2.21), (2.22) we easily obtain that

$$\widehat{u_1}(p, t) - \widehat{u_2}(p, t) = \int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)[\widehat{f}_{v_1}(p, s) - \widehat{f}_{v_2}(p, s)]ds. \tag{2.23}$$

Clearly, the upper bound on the norm

$$\|\widehat{u_1}(p, t) - \widehat{u_2}(p, t)\|_{L^2(\mathbb{R})} \leq \int_0^t \|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)[\widehat{f}_{v_1}(p, s) - \widehat{f}_{v_2}(p, s)]\|_{L^2(\mathbb{R})}ds \tag{2.24}$$

holds. Let us use inequality (1.4) to derive the upper bound as

$$\begin{aligned}
&\|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)[\widehat{f}_{v_1}(p, s) - \widehat{f}_{v_2}(p, s)]\|^2_{L^2(\mathbb{R})} \\
&= 2\pi \int_{-\infty}^{\infty} e^{-2(t-s)p^4} e^{2(t-s)a}|\widehat{G}(p)|^2|\widehat{f}_{v_1}(p, s) - \widehat{f}_{v_2}(p, s)|^2 dp \\
&\leq 2\pi e^{2aT}\|\widehat{G}(p)\|^2_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} |\widehat{f}_{v_1}(p, s) - \widehat{f}_{v_2}(p, s)|^2 dp \\
&\leq e^{2aT}\|G(x)\|^2_{L^1(\mathbb{R})}\|F(v_1(x, s), x) - F(v_2(x, s), x)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Recalling formula (1.12) gives us

$$\|F(v_1(x,s),x) - F(v_2(x,s),x)\|_{L^2(\mathbb{R})} \le l\|v_1(x,s) - v_2(x,s)\|_{L^2(\mathbb{R})}, \tag{2.25}$$

so that

$$\begin{aligned} &\|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}\widehat{G}(p)[\widehat{f}_{v_1}(p,s) - \widehat{f}_{v_2}(p,s)]\|_{L^2(\mathbb{R})} \\ &\quad \le e^{aT} l\|G(x)\|_{L^1(\mathbb{R})}\|v_1(x,s) - v_2(x,s)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{2.26}$$

By means of (2.24) along with (2.26),

$$\begin{aligned} &\|\widehat{u_1}(p,t) - \widehat{u_2}(p,t)\|_{L^2(\mathbb{R})} \\ &\quad \le e^{aT} l\|G(x)\|_{L^1(\mathbb{R})} \int_0^T \|v_1(x,s) - v_2(x,s)\|_{L^2(\mathbb{R})} ds. \end{aligned}$$

According to the Schwarz inequality

$$\int_0^T \|v_1(x,s) - v_2(x,s)\|_{L^2(\mathbb{R})} ds \le \sqrt{\int_0^T \|v_1(x,s) - v_2(x,s)\|^2_{L^2(\mathbb{R})} ds}\sqrt{T}. \tag{2.27}$$

Hence,

$$\begin{aligned} &\|\widehat{u_1}(p,t) - \widehat{u_2}(p,t)\|_{L^2(\mathbb{R})} \\ &\quad \le e^{aT} l\sqrt{T}\|G(x)\|_{L^1(\mathbb{R})}\|v_1(x,t) - v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])}. \end{aligned} \tag{2.28}$$

We arrive at

$$\begin{aligned} \|u_1(x,t) - u_2(x,t)\|^2_{L^2(\mathbb{R}\times[0,T])} &= \int_0^T \|\widehat{u_1}(p,t) - \widehat{u_2}(p,t)\|^2_{L^2(\mathbb{R})} dt \\ &\le e^{2aT} l^2 T^2 \|G(x)\|^2_{L^1(\mathbb{R})}\|v_1(x,t) - v_2(x,t)\|^2_{L^2(\mathbb{R}\times[0,T])}. \end{aligned} \tag{2.29}$$

Let us recall (2.23). Thus,

$$p^4[\widehat{u_1}(p,t) - \widehat{u_2}(p,t)] = \int_0^t e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}p^4\widehat{G}(p)[\widehat{f}_{v_1}(p,s) - \widehat{f}_{v_2}(p,s)]ds.$$

It is not difficult to see that the norm can be estimated from above as

$$\begin{aligned} &\|p^4[\widehat{u_1}(p,t) - \widehat{u_2}(p,t)]\|_{L^2(\mathbb{R})} \\ &\quad \le \int_0^t \|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}p^4\widehat{G}(p)[\widehat{f}_{v_1}(p,s) - \widehat{f}_{v_2}(p,s)]\|_{L^2(\mathbb{R})} ds. \end{aligned} \tag{2.30}$$

By virtue of inequality (1.5) we obtain the estimate from above

$$\|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}p^4\widehat{G}(p)[\widehat{f}_{v_1}(p,s)-\widehat{f}_{v_2}(p,s)]\|^2_{L^2(\mathbb{R})}$$

$$= 2\pi\int_{-\infty}^{\infty} e^{-2(t-s)p^4}e^{2(t-s)a}|p^4\widehat{G}(p)|^2|\widehat{f}_{v_1}(p,s)-\widehat{f}_{v_2}(p,s)|^2dp$$

$$\leq 2\pi e^{2aT}\|p^4\widehat{G}(p)\|^2_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}|\widehat{f}_{v_1}(p,s)-\widehat{f}_{v_2}(p,s)|^2dp$$

$$\leq e^{2aT}\left\|\frac{d^4G}{dx^4}\right\|^2_{L^1(\mathbb{R})}\|F(v_1(x,s),x)-F(v_2(x,s),x)\|^2_{L^2(\mathbb{R})}.$$

Using formula (2.25), we derive

$$\|e^{(t-s)\{-p^4+ibp+a\}}\sqrt{2\pi}p^4\widehat{G}(p)[\widehat{f}_{v_1}(p,s)-\widehat{f}_{v_2}(p,s)]\|_{L^2(\mathbb{R})}$$
$$\leq e^{aT}l\left\|\frac{d^4G}{dx^4}\right\|_{L^1(\mathbb{R})}\|v_1(x,s)-v_2(x,s)\|_{L^2(\mathbb{R})}. \tag{2.31}$$

Let us combine bounds (2.30), (2.31) and (2.27). This yields

$$\|p^4[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|_{L^2(\mathbb{R})}$$
$$\leq e^{aT}\sqrt{T}l\left\|\frac{d^4G}{dx^4}\right\|_{L^1(\mathbb{R})}\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])}. \tag{2.32}$$

Therefore,

$$\left\|\frac{\partial^4}{\partial x^4}[u_1(x,t)-u_2(x,t)]\right\|^2_{L^2(\mathbb{R}\times[0,T])} = \int_0^T\|p^4[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|^2_{L^2(\mathbb{R})}dt$$
$$\leq e^{2aT}l^2T^2\left\|\frac{d^4G}{dx^4}\right\|^2_{L^1(\mathbb{R})}\|v_1(x,t)-v_2(x,t)\|^2_{L^2(\mathbb{R}\times[0,T])}. \tag{2.33}$$

By means of (2.23), we have

$$\frac{\partial}{\partial t}[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]$$
$$=\{-p^4+ibp+a\}[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]+\sqrt{2\pi}\widehat{G}(p)[\widehat{f}_{v_1}(p,t)-\widehat{f}_{v_2}(p,t)].$$

Hence,

$$\left\|\frac{\partial}{\partial t}[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\right\|_{L^2(\mathbb{R})}\leq a\|\widehat{u_1}(p,t)-\widehat{u_2}(p,t)\|_{L^2(\mathbb{R})}$$
$$+|b|\|p[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|_{L^2(\mathbb{R})}+\|p^4[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|_{L^2(\mathbb{R})}$$
$$+\sqrt{2\pi}\|\widehat{G}(p)[\widehat{f}_{v_1}(p,t)-\widehat{f}_{v_2}(p,t)]\|_{L^2(\mathbb{R})}. \tag{2.34}$$

It is not difficult to see that according to (2.28), the first term in the right side of (2.34) can be estimated from above by

$$age^{aT}\sqrt{T}l\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])} \tag{2.35}$$

with g is introduced in (1.6). We derive the upper bound on the norm as

$$\|p[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|^2_{L^2(\mathbb{R})}$$
$$=\int_{|p|\le 1} p^2|\widehat{u_1}(p,t)-\widehat{u_2}(p,t)|^2dp+\int_{|p|>1} p^2|\widehat{u_1}(p,t)-\widehat{u_2}(p,t)|^2dp$$
$$\le \|\widehat{u_1}(p,t)-\widehat{u_2}(p,t)\|^2_{L^2(\mathbb{R})}+\|p^4[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\|^2_{L^2(\mathbb{R})}.$$

Let us use inequalities (2.28) and (2.32). They give the estimate from above on the second term in the right side of (2.34) equal to

$$|b|ge^{aT}\sqrt{T}l\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])}. \tag{2.36}$$

By virtue of (2.32), the third term in the right side of (2.34) can be bounded from above by

$$ge^{aT}\sqrt{T}l\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])}. \tag{2.37}$$

By means of (1.4) and (2.25), we obtain that

$$2\pi\int_{-\infty}^{\infty}|\widehat{G}(p)|^2|\widehat{f}_{v_1}(p,t)-\widehat{f}_{v_2}(p,t)|^2dp$$
$$\le 2\pi\|\widehat{G}(p)\|^2_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}|\widehat{f}_{v_1}(p,t)-\widehat{f}_{v_2}(p,t)|^2dp$$
$$\le \|G(x)\|^2_{L^1(\mathbb{R})}\|F(v_1(x,t),x)-F(v_2(x,t),x)\|^2_{L^2(\mathbb{R})}$$
$$\le \|G(x)\|^2_{L^1(\mathbb{R})}l^2\|v_1(x,t)-v_2(x,t)\|^2_{L^2(\mathbb{R})}.$$

Thus, the fourth term in the right side of (2.34) can be estimated from above by

$$gl\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R})}. \tag{2.38}$$

Let us combine (2.35), (2.36), (2.37) and (2.38). This yields

$$\left\|\frac{\partial}{\partial t}[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\right\|_{L^2(\mathbb{R})}$$
$$\le ge^{aT}\sqrt{T}l\{a+|b|+1\}\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R}\times[0,T])}+gl\|v_1(x,t)-v_2(x,t)\|_{L^2(\mathbb{R})}.$$

We arrive at the upper bound on the norm as

$$\left\|\frac{\partial}{\partial t}(u_1(x,t)-u_2(x,t))\right\|^2_{L^2(\mathbb{R}\times[0,T])}=\int_0^T\left\|\frac{\partial}{\partial t}[\widehat{u_1}(p,t)-\widehat{u_2}(p,t)]\right\|^2_{L^2(\mathbb{R})}dt$$
$$\le 2g^2l^2[e^{2aT}T^2\{a+|b|+1\}^2+1]\|v_1(x,t)-v_2(x,t)\|^2_{L^2(\mathbb{R}\times[0,T])}. \tag{2.39}$$

We recall the definition of the norm (1.10) and use inequalities (2.29), (2.33) and (2.39). It is not difficult to see that a calculation gives us

$$\|u_1-u_2\|_{W^{(1,4),2}(\mathbb{R}\times[0,T])}$$
$$\le gl\sqrt{T^2e^{2aT}(1+2[a+|b|+1]^2)+2}\|v_1-v_2\|_{W^{(1,4),2}(\mathbb{R}\times[0,T])}. \tag{2.40}$$

The constant in the right side of (2.40) is less than one via inequality (1.17). This means that under the stated assumptions problem (1.16) defines the map

$$\tau_{a,b}:W^{(1,4),2}(\mathbb{R}\times[0,T])\to W^{(1,4),2}(\mathbb{R}\times[0,T]),$$

which is a strict contraction. Its unique fixed point $w(x,t)$ is the only solution of problem (1.1), (1.2) in $W^{(1,4),2}(\mathbb{R}\times[0,T])$. $\square$

Proof of Corollary 1.4. The validity of the statement of our Corollary follows from the fact that the constant in the right side of bound (2.40) is independent of the initial condition (1.2) (see, e.g., [17]). Therefore, problem (1.1), (1.2) admits a unique solution $w(x,t)\in W^{(1,4),2}(\mathbb{R}\times\mathbb{R}^+)$. We suppose that $w(x,t)$ is trivial for $x\in\mathbb{R}$ and $t\in\mathbb{R}^+$. This will contradict to the assumption that $\text{supp}\widehat{F(0,x)}\cap\text{supp}\widehat{G}$ is a set of nonzero Lebesgue measure on $\mathbb{R}$. $\square$

Acknowledgements

V.V. is grateful to Israel Michael Sigal for the partial support by the NSERC grant NA 7901.

Author contributions Messoud Efendiev and Vitali Vougalter wrote the manuscript. All authors reviewed the manuscript.

Funding The work was partially supported by the NSERC grant NA 7901.

Data Availability Statement No datasets were generated or analyzed during the current study.

Declarations

Ethical approval The results of the work are applicable for both human and animal studies.

Publishing policy The authors have read and understood the publishing policy and submit this manuscript in accordance with this policy.

Conflict of interest policy The authors declare that they have no conflict of interest as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Dual publication The results in this manuscript have not been published elsewhere, nor are they under consideration by another publisher.

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(Received: March 20, 2025; revised: July 6, 2025; accepted: September 27, 2025)