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RESEARCH ARTICLE

Phase Transitions in an Ising Model of Agent Expectations in Financial Markets: Analytics and Numerical Results in One and Two-Dimensional Network Topologies

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ABSTRACT We cite correspondences between dynamics in competitive markets and information theory in the objective of recovering signal from noisy information sequences. In financial markets, this objective has been examined as recovering signal on phase transitions between ordered and disordered states of agents in the market. These transitions have been indicated to denote critical points in time series of market price. Although there is a noteworthy background in information theory in the study of the dynamics of the Ising model in this manuscript, we pursue a different modeling approach. Whereas phase transitions in a multicomponent model of market states have previously been studied with numerical methods, we provide an analytical demonstration that a multicomponent model as an Ising analogue can evidence phase transitions.

INDEX TERMS Agent price expectations, Ising market models, multicomponent models of phase transitions, signal in financial markets.

I. INTRODUCTION

Multicomponent Ising analogue models continue to have diverse engineering applications that include information storage [1], dynamics of quantum phase transitions [2], and combinatorial optimization problems [3]. Applications to social processes include opinion formation in networks ([4], [5]). A number of recent engineering applications have recognized the contributions that Ising analogue models can offer to the analysis of financial markets (e.g., [6], [7], [8], [9]). Most recently, [8] have enacted a price prediction Ising analogue model based on Deep Learning of investor sentiment. Even if the cited investigators do not specify their model as Ising, they do address multicomponent models that a range of authors have directly considered as Ising analogs since they represent intrinsic spin (i.e., the internal spin valence of an agent) and spin valences of agents in the field within which the agent is integrated. See, for example, Ecrot et al. ([7]), Lima et al. ([10]), and Takaishi [11].

Intuitively, the internal dynamics of spins and local influence of neighbors and near neighbors in Ising analogues have a correspondence to the influence of rationality (e.g., Sargent 2013 [12]) and of neighboring entities. See [13], [14] on "herding" in financial markets. A normative objective in the models has been in the recovery of signal from "noisy" information sequences. Ising analogues offer natural representation of phase transitions between ordered and disordered states in the dynamics of a market that are considered to denote critical points in market cycles (e.g., [15]). In the model that we propose, ordered states can be considered to result from the "bounded" rational processing of an agent. Disordered states occur when the influence of

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other agents (i.e., ‘‘herding’’) dominates. Both states have been well-cited in financial markets [4], [16].

Across the range of available applications, Ising model analogues have been solved numerically. However, analytical solutions remain of interest since they can contribute to the validation and calibration of numerical solutions. Onsager's [17] classic work provided a one-dimensional (1D) analytic solution to Ising's model [18]. Moreover, in Fenz et al. [19], the authors, using integral equation theory, develop a framework that incorporates correlations and allows quantitative predictions of the phase behavior of Ising mixtures, as opposed to previously used methods that would allow only qualitative comparisons. Furthermore, in Julius and Teller [20], the authors consider a 2D square net (lattice) with four kinds of atoms where only nearest neighbors interact and there exist two potential energies, namely one between like and one between unlike atoms. By means of extension of Onsager's methods, the authors demonstrate analytically that long range order exists at low temperatures if the like atoms attract but not when they repel each other. Finally, Halperin [21] models the approximate return of the market index of a heterogeneous market via mean field of a homogeneous market comprising the large number of replicas of the representative stock with the self-interaction potential. This results in the McKean-Vlasov equation governing the dynamics of the system, which is essentially a form of a non-linear Fokker-Planck equation. This, in turn, results in complex dynamics yielding ergodicity breaking and phase transitions in different parameter regimes. The analysis of the phase structure of the model under parameter variations is based on the underlying self-consistency equation. Finally, a second order phase transition emerges in the case of symmetric potential while the first order phase transition occurs in the case of asymmetric potential. In the discourse to follow, we will propose analytics of both 1D and two-dimensional (2D) solutions to the Ising analogue multicomponent model as applied to agents in a financial market.

II. THEORY

A. PHASE TRANSITIONS IN MULTICOMPONENT SYSTEMS

Phase transitions between ordered and disordered states are among the most cited properties of 2D Ising models (e.g., [22]). In applications to financial markets, phase transitions have been reported to commonly occur at critical points in the dynamics of price (e.g., [23], [24], [25]) and this increases their importance to market dynamics. Indeed, significant developments over the past few decades have been achieved in the study of Ising models of financial dynamics within the Econophysics literature (Sornette [26], Johansen et al. [27] and Cividino et al. [28]. Sornette (2014) [26] demonstrates that a significant number of economic models can be mapped into different versions of the Ising model, thereby capturing the effect of social interactions on individual decision making. This investigator provides illustrative examples of the endogenous price formation mechanism as a function of previous price, cumulative decisions on buying and selling performed by individual agents and the exogenous randomness are presented. More specifically still, the formula for the time-varying behavior of each agent is provided as a function of the underlying network topology by means of solving the profit maximization problem. Two important parameters are found to be driving the dynamics of the system, namely the tendency towards imitation and the tendency towards idiosyncratic behavior. The existence of the underlying Ising phase transition was found to lead to the emergence of collective imitation that translates into crashes in the financial system. Moreover, in Johansen et al. [27], the authors study a rational expectation model of bubbles and crashes. They provide strong evidence for the claim that the stock market crashes are caused by the slow build-up of long-range correlations leading to the collapse of the stock market in one critical instant. It is assumed that each trader can either buy or sell and that the state of the trader is modeled by a class of stochastic dynamical models of interacting particles. The two key parameters of the system are, as in Sornette (2014) [26], the tendency towards imitation κ, the tendency towards the idiosyncratic behavior σ together with the global influence term G. These parameters together capture agent market behavior and thus advocate for the importance of behavioral modeling, the absence of which results in models unable to explain certain empirical regularities such as bubbles and crashes. The susceptibility of the system is the derivative (with respect to G) of the expected value of M, which stands for the average state. Moreover, the hazard rate corresponding to the random variable which stands for the time of crash is assumed to behave in the same way as the susceptibility, which is in turn used to derive the formula for the market price before the crash. The corresponding formula is derived in case of different topologies including the 2D grid and the hierarchical diamond lattice. Upon calibration, the model which assumes the hierarchical diamond lattice representation of the networked agents demonstrates impressive performance as it was able to predict the timing of the 1987 stock market crash with the error margin of just several days. Equally impressive is the fact that the model managed to predict the other major market crashes including the ones in 1929 and 1997. Finally, the model is reported to outperform state-of-the-art econometric models such as the GARCH (1,1) model. Finally, Cividino et al. [28] introduce an agent-based model where the market consists of two types of traders, namely the fundamentalists and the noise traders. There are a number of risky assets and one riskless asset available. Risky assets pay stochastic dividends modelled via multidimensional Brownian motion with specified mean and covariance matrix. The fundamentalists invest portions of their income into risky and riskless assets so as to maximize the expected utility of their wealth and the authors provide the corresponding closed form formula. For noise traders, the spin vector is introduced to represent their portfolio allocation. To model the trend-following attitude of noise traders, a vectoral external field is introduced corresponding to the price momentum. The interactions between spins are defined via the specified Hamiltonian while a given configuration of spins is assumed to follow the standard Boltzman distribution. Importantly, this transition probability is a function of parameter κ which quantifies the herding capacity. These ideas lead to the derivation of the formula for transition probability for the update of portfolio allocations for the noise traders. The market price is computed numerically by means of an iterated scheme and is assumed to be consistent with the Walrasian auction. The authors, by means of simulations, demonstrate the existence of a critical value κ_c which drives the system dynamics. More specifically, if $\kappa < \kappa_c$, no super-exponential bubbles occur, if $\kappa \approx \kappa_c$ the emergence of well-defined bubbles is reported while for $\kappa > \kappa_c$ the system exhibits clustering or cascades of bubbles. Finally, the resulting market price time series, on top of the ability to explain the emergence of bubbles and crashes, equally reproduces several known properties of financial markets such as volatility clustering and fat tails of the return distribution. The bubble is shown to emerge once noise traders polarize their opinions towards more than one assay. Additional noteworthy contribution in this direction can be found in (e.g. [7], [29], [30], [31], [32]). We address agent expectations for price in financial markets that are well recognized as predictors of market price [12].

We will first propose a multi-component model of agent expectations as signal on market price. We will then follow stylized methodology of Onsager [17] in an Ising model and analytically demonstrate that the multicomponent model can evidence phase transitions between ordered and disordered states of agents. Whereas Onsager [17] demonstrates order in terms of a positive correlation in states of what in our model is network agents, our results extend this definition by showing that any two agents in our model will tend to a perfect correlation in a 2D case and its equivalent in a 1D case. Moreover, perfect correlation occurs almost instantly as the convergence rate is demonstrated to be exponentially fast. These results follow a method of limits in a case in which interactions have run an adequate number of iterations for convergence. Finally, computer simulations show that the results in one-dimensional case are most possible within the assumed model set-up.

B. MULTICOMPONENT MODEL IN A FINANCIAL MARKET

As now well-recognized, agents in financial markets exhibit both components of individually-based processing and collective influences in their price expectations (e.g., [24]). We agree with previous investigators that (1) agent expectations are likely to be generated by simultaneous processing in components that have different dynamics and (2) phase transitions are inherent in the dynamics as applied to markets [29], [33].

The model presented in this manuscript directly follows from a detailed elaboration on distinctive behavioral foundations of agent microprocessing and is continuous in expectations rather than in binary states.

C. BOUNDED RATIONAL PROCESSING

The designation of bounded rationality by Nobel Laureate Herbert Simon allows for imperfect implementation of rationality with a correction based on learning from an agent's history in prediction from fundamentals of value. The first term of the multicomponent model that we propose in (1) represents an indicator of intrinsic or fundamental value, i.e., the value as indicated by metrics such as price-earnings (PE) or price-earning to growth (PEG) ratios cash flow, and debt to equity ratios. The reason why agents tend to have common projections is that they share metrics that include PE and PEG ratios. This is not to say that such common projections necessarily indicate efficient markets. Additionally, reference groups largely based on demographics can be reasons why agents have commonality. The second term in (1) gives a form to agent learning from their histories. It corrects for errors in past predictions from the metrics of fundamental value.

The concept of bounded rationality that we invoke is consistent with Simon's designation but differ in several aspects from Hegselmann and Krause's invocation ([34]). For these authors, bounded confidence is a way of imposing the neighbor structure: an agent will only be influenced by those agents whose initial beliefs/opinions do not differ in absolute value more than a specified agent-specific margin. In other words, the impact of any other agent on the specified agent will be 0 if their initial opinions are sufficiently far apart. In a fixed network structure that we assign, agents are drawn independently from the same distribution on the interval [0,10000]. As such, the configuration of neighbors is completely disjoint from the agents' initial expectation levels which initiate the run of the model. Furthermore, in our case, the term "bounded rational" simply refers to the first component of the agents' expectations. In this component, the update mechanism is considered (boundedly) rational as agent only learns from her own previous projections (i.e. from previous experiences and exhibited errors). The second component corresponds to herding behavior, that is the imitation of the behavior of others which is largely considered to be irrational in the economic literature.

$$\epsilon_{j,t+1}^{cog} = \overline{V_t} + \sum_{\theta=1}^{\tau}(\overline{V_{t-\theta}} - \overline{P_{t-\theta}}) \tag{1}$$

where ϵ_j^{cog} is the bounded rational component of expectations of the j^{th} agent, $\overline{V_t}$ is the mean of a vector of market indicators of the fundamental value of an asset in a defined time interval and $\overline{P_t}$ is a market price for an asset or grouping of assets in a defined time interval.

D. PROCESSING IN HERDING BEHAVIOR

The second term of the multicomponent model represents the exogenous influence of network neighbors on agent expectations. A number of multicomponent models of expectations recognize the sensitivity of levels of an agent's expectations to deviations from the levels of cohorts or network neighbors as "herding". Several investigators designate the sensitivity in "herding" as emotion-based [16]). This is in contrast to the first component of equation 1. In the first component, as noted, we follow Simon and designate processing in this component as bounded rational and thereby cognitive. Agents are considered to be more sensitive to deviation from norms in emotion-based processing than in cognitive processing. Consistent with this in the case of emotion-based processing, the distance of an agent's distance from the level of his or her network neighbors increases exponentially in its influence on an agent's expectations. Following this literature, we give a polynomial form to the influence of deviations from an agent's cohorts or network neighbors to his/her expectations. The β parameter denotes the importance of differences of an i^{th} level of expectation from the j^{th} level. It is given an exponential form since literature indicates the sensitivity of an agent's expectations to the expectations of other agents.

In the term for a component that can generate "herding" behavior, we follow a range of investigators in representing an inherent tendency of economic agents in a network to align themselves with the prevalent orientations of network neighbors either as inferred from observations of the market behavior of neighbors or a proxy such as word-of-mouth or expectations. In (2), the sign and magnitude of processing that generates herding are defined by the difference between own expectations and those of neighbors across time periods relative to a unit of dispersion.

$$\epsilon^{aff}_{j,t+1} = \sum_{i=1}^{n} M_{ij}(\epsilon^{t}_{j} - \epsilon_{i,t})^{\beta}/\sigma \tag{2}$$

where $\beta \geq 1$. In (2) ϵ is the level of positive or negative expectation for a given price whereas β is an accelerator parameter in response functions that can represent the differences in the magnitude of increments to herding-based processing in comparison to increments in bounded rational processing. M_{ij} is the weight of the interpersonal influence of agent j on agent i. The parameter σ in (2) is the standard deviation of the distribution of expectations across neighbors, whereas ϵ^{aff}_{j} is the sentiment level of the j^{th} agent $i \neq j$ and n is the number of agents in a network.

Combining (1) and (2) in (3) gives an elaborated form of a 2D Ising model of expectations in bounded rational and herding-generating components:

$$\epsilon^{i}_{t} = \alpha_t(\overline{V_t} + \sum_{\theta=1}^{\tau}(\overline{V_{t-\theta}} - \overline{P_{t-\theta}})) + \frac{1-\alpha_t}{\sigma}(\sum_{i=1}^{n} M_{ij}(\Delta_{j,i}(t))^{\beta}) \tag{3}$$

where $0 \leq \alpha_t \leq 1$ and $1 - \alpha_t$ are the weights of the components in overall expectations, $\Delta_{i,j}(t) = \epsilon^{i}_{t} - \epsilon^{j}_{t}$ and other variables are as previously defined. In (3), we define α_t as a time-varying weight function that depends on the distance between market price and fundamental value in a transition function. In the next section, we will elaborate on the behavioral basis for dynamics in the transition function, α_t. Based on principles from Quantum Mechanics, more specifically Fokker-Planck and Schrödinger equations, the authors of ([35]) analytically demonstrate that the stock returns follow power-law distribution which they fully parameterize in closed form. They build on the work of previous investigators that has demonstrated that the power-law exponent can be used as a proxy for the strength of the herding behavior, namely, the larger the value of the exponent the weaker the herding. Using theoretical derivations developed in the manuscript, the authors subsequently estimate the power-law exponent from the data. They subsequently use this information to establish statistically significant relationship between both herding behavior and periods of high and low economic output and between economic uncertainty and counter-cyclical herding behavior.

E. TRANSITIONS IN COMPONENT DOMINANCE

The relative influence of bounded rational and herd-based processing in multi-component dynamics that we define can be given a form in a transition function for component weights. A basis for an agent's processing that underlies non-linear dynamics in the transition function is offered by theorists who are sometimes labeled as incongruity theorists in expectations (e.g., [36], [37]). These theorists maintain that, whereas small violations of in-place expectations generate interest and an attempt to explain and reconcile the violations (as in bounded rational processing), larger violations increasingly generate fear and "trepidation" as in a category of processing that is emotion-based (e.g., [38]). In such processing, agents anchor to levels in the processing of others. Stylized micro-processing in the dynamics of a transition function for the relative influence of components that we describe can be qualitatively described in the graphic of Fig. 1

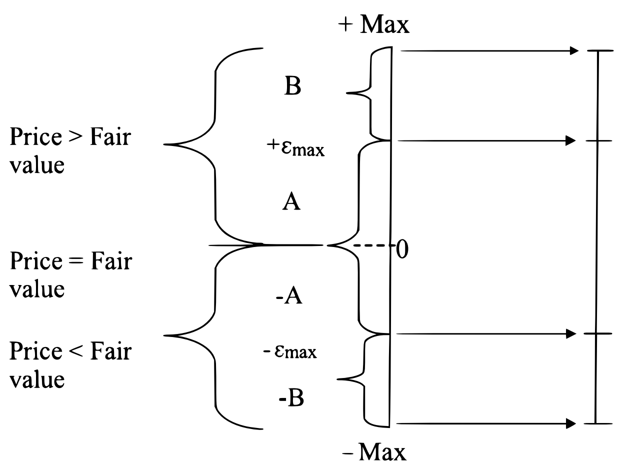

FIGURE 1. Relationship between market price and fair value.

Fig. 1 assumes a symmetrical positive and negative range of expectations for the price of an asset for which there is a market large enough to be efficient in matching buyers and sellers. At a mid-point of the range ϵ_0, expectations are assumed to be equal to the price implied by fundamental value as commonly defined by metrics of intrinsic value since there is no difference between market price and metrics of fundamental value (i.e., $\epsilon_0 = \overline{V_0}$). Even in what we designate as bounded rational processing, there can be more or less random variation around fundamental value as in the defined region of A. The levels of $-\epsilon_{max}$ or $+\epsilon_{max}$ denoted as the boundary of region A can be considered as the boundary of a confidence interval in which a bounded rational component dominates. Price variation outside of this region increasingly generates emotional biases in agent processing. These representations capture the fact that agents' behaviour is indeed strategic since they, by means of adjusting the relative influence of bounded rational and herd-based expectations component, learn from their previous experiences and adapt to the changing envirnoment.

F. A STOCHASTIC MODEL OF AGENT MULTI-COMPONENT EXPECTATIONS

To support the instantiations of an Ising model in a financial market that we have elaborated, we next implement a stochastic model of dynamics of market price in the first term of (3). Stochastic dynamics have been invoked in a long history of giving a form to processes that generate market price (e.g., [39], [40], [41]).

We recognize that when the asset is an indexed bundle of equities, prices have historically grown linearly with local perturbations some of which attain a magnitude to be classified as a cycle. In (4), we follow [39], [41], and represent the price in short intervals as a stochastic process in which there are random perturbations from a linear trend. Consistent with well-cited background models and a range of more recent specifications (e.g., [42], [43], [44], [45]) this price dynamic can be given a simple representation as Brownian motion with drift.

$$\epsilon_t^i = \alpha_t(\overline{V_t} + \sum_{\theta=1}^{\tau}(\overline{V_{t-\theta}} - \overline{P_{t-\theta}})) + (1-\alpha_t)\sum_{i=1}^{n} M_{ij}\Delta_{i,j}(t) \tag{4}$$

Here ϵ_i represents an i^{th} agent's expectation for an asset price, α_t is the weight of the component that can be designated as bounded rational in an agent's expectations, V_t is a predictor of next period market price that depends on metrics of fundamental value, P_t is Brownian motion with drift, $1-\alpha_t$ is the weight of the processing component in which generates in "herding" and M_{ij} is the weight of the interpersonal influence of agent j on agent i. We can re-write P in (4) as

$$P_t = \gamma t + \delta W_t$$

where γ is drift and δ is the volatility of Brownian motion. From the Law of the Iterated Logarithm for Brownian motion (e.g. [44], [46], [47]), it follows that:

$$\lim_{t\to\infty} \frac{W_t}{(2t\log\log t)^{\frac{1}{2}}} = 1 \tag{5}$$

with probability one. If we assume that all agents can rationally predict the stock price in the long run, the Law of the Iterated Logarithm allows the further inference that $V_t = \gamma t$ as the Brownian fluctuations become negligible in the limit. Local perturbations do remain of interest in any finite time. In the next section, we will analytically demonstrate that the multicomponent model we propose can demonstrate the phase transitions that are a fundamental property of a 2D Ising model across a range of applications.

G. EMPIRICAL SUPPORT

To support a multicomponent model of the form we propose this manuscript, Silver and Raseta ([48]) demonstrate effects in both time series and cross-sectional data of the University of Michigan's Index of Consumer Sentiment, the most commonly cited measure of expectations. The cross-sectional data is given a form that represents an agent's distance in expectations from the mean of his or her cohorts. The designation of cohorts is in demographic indicators that are provided in the cross-sectional data. Silver and Raseta ([48]) demonstrate significant effects of an agent's distance from a cohort's level in the cross-section on his or her level of expectations as well as agency effects in the time series.

H. PHASE TRANSITIONS IN THE EXPECTATIONS OF NETWORKED AGENTS

We follow [17] often cited definition in demonstrating phase transition between ordered and disordered states in a 2D Ising model.

I. BACKGROUND

Definition 1: Let $\mathbb{G}$ be a simple graph on n vertices. Let D be its degree matrix, that is $D_{ij} = \delta_{ij} \times deg(v_i)$, where $deg(v_i)$ is the degree of the vertex v_i and δ_{ij} is the Kronecker symbol. Moreover, let A be the adjacency matrix of $\mathbb{G}$ that is $A_{ij} = 1$ if i and j are neighbors whereas all the other elements of the matrix are null. Then the Laplacian matrix associated to $\mathbb{G}$ is an $n \times n$ matrix L defined by $L = D - A$.

Theorem 1: Following Shu et al. [49] let $\mathbb{G}$ be a connected graph with n vertices. The largest eigenvalue of the Laplacian matrix of $\mathbb{G}$ is denoted by $\mu(\mathbb{G})$. Suppose the degree sequence of $\mathbb{G}$ is $d_1 \geq d_2 \cdots \geq d_n$. Then

$$\mu(\mathbb{G}) \leq d_n + \frac{1}{2} + \left((d_n - \frac{1}{2}) + \sum_{i=1}^{n} d_i(d_i - d_n)\right)^{\frac{1}{2}} \tag{6}$$

III. RESULTS

A. ANALYTIC RESULTS

Theorem 2: Let $N \geq 3$ be the number of agents on the line with the classical wrap-around making sure all agents have exactly 2 neighbors. Let

$\Phi_t = V_t - P_t$

and furthermore, assume that the agents' expectations evolve according to the following formulae:

$$\epsilon_t^i = \alpha_t(V_t + \Phi_{t-1}) + (1-\alpha_t)\sum_j M\Delta_{j,i}(t-1)$$
$$\alpha_t = \begin{cases} \alpha_1, & \text{if } |\Phi(t)| \leq K_t^{\frac{1}{2}} \text{ or } |\Phi(t)| \geq 2K_t^{\frac{1}{2}} \\ 1-\alpha_1, & \text{otherwise} \end{cases}$$
$$P_t = \gamma t + \delta W_t \tag{7}$$

where W_t is a standard Brownian motion defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $K_t := \mathbb{V}ar(P_t)$ and α_t is a time-varying stochastic weight function that depends on the distance between market price and fundamental value. Moreover, let $\alpha^* = \max(\alpha_1, 1-\alpha_1)$

Then, for all $M \in (0, \frac{1}{4\alpha^*})$

$\mathbb{C}orr(\epsilon_t^i, \epsilon_t^j) \rightarrow 1$ as $t \rightarrow \infty$

for all pairs of agents i and j.

Proof: Define the random process $C_t := M\xi_t$. Simple algebra then yields:

$$\begin{aligned} \Delta_{1,2}(t+1) &= M\xi_t(2\Delta_{1,2}(t) - \Delta_{N,1}(t) - \Delta_{2,3}(t)) \\ \Delta_{2,3}(t+1) &= M\xi_t(2\Delta_{2,3}(t) - \Delta_{1,2}(t) - \Delta_{3,4}(t)) \\ &\dots \\ \Delta_{N,1}(t+1) &= M\xi_t(2\Delta_{N,1}(t) - \Delta_{1,2}(t) - \Delta_{N-1,N}(t)) \end{aligned} \tag{8}$$

where $\xi_t := 1-\alpha_t$

Define an N-dimensional vector $\Omega_\mathbf{t}$ by:

$$\Omega_\mathbf{t} = \begin{bmatrix} \Delta_{1,2}(t) \\ \Delta_{2,3}(t) \\ \vdots \\ \Delta_{N-1,N}(t) \\ \Delta_{N,1}(t) \end{bmatrix} \tag{9}$$

Define an $N \times N$ matrix L by:

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ -1 & 0 & \dots & & -1 & 2 \end{bmatrix} \tag{10}$$

The above system exhibits the following solution in its vector representation:

$$\Omega_\mathbf{t} = M^t \prod_{j=1}^{t}(\alpha_j - 1)L^t\Omega_\mathbf{0} \tag{11}$$

Observe that L is a real symmetric matrix and hence L is diagonalizable by an orthogonal matrix and thus the above can be rewritten as:

$$\Omega_\mathbf{t} = M^t \prod_{j=1}^{t}(\alpha_j - 1)S^T \begin{bmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_N^t \end{bmatrix} S\Omega_\mathbf{0} \tag{12}$$

where λ_i are the eigenvalues of L.

Let $\mathbb{G}$ stand for the graph whose vertices and edges capture the topology of agents on the line with the classical wrap-around.

It can be seen that L is exactly the Laplacian matrix of $\mathbb{G}$. Observe that $d_i = 2$ for all $i \in 1, \dots, N$ and thus $\mu(\mathbb{G}) \leq 4$ for all N simultaneously. It is well known that L is a positive semi-definite matrix. This immediately gives a simultaneous upper bound for all components l of $\Omega_\mathbf{t}$:

$$|\Omega_\mathbf{t}{}^l| \leq C(N)(4\alpha^*)^t \tag{13}$$

where $C(N)$ is a constant that depends on N alone. In other words:

$$|\Delta_{i,j}(t)| \leq C(N)(4\alpha^*)^t \tag{14}$$

for any two agents i and j. Now let A and B stand for any two agents in the network. We will now show that

$\mathbb{C}orr(A_t, B_t) \rightarrow 1$ as $t \rightarrow \infty$

Towards this end, let $\Delta_{AB}(t) := A_t - B_t$. Then:

$$\mathbb{C}orr(A_t, B_t) = \frac{\mathbb{E}(A_tB_t) - \mathbb{E}(A_t) \times \mathbb{E}(B_t)}{(\mathbb{V}ar(A_t) \times \mathbb{V}ar(B_t))^{\frac{1}{2}}} \tag{15}$$

Observe that then:

$$Numerator = \mathbb{V}ar(A_t) - \mathbb{E}(A_t\Delta_{AB}(t)) + \mathbb{E}(A_t)\mathbb{E}(\Delta_{AB}(t)) \tag{16}$$

$$\begin{aligned} \mathbb{V}ar(B_t) = &\ \mathbb{V}ar(A_t) - 2\mathbb{E}(A_t\Delta_{AB}(t)) \\ &- 2\mathbb{E}(A_t)\mathbb{E}(\Delta_{AB}(t)) + \mathbb{E}\Delta_{AB}^2(t) - (\mathbb{E}\Delta_{AB}(t))^2 \end{aligned} \tag{17}$$

Using the Law of the Iterated Logarithm for Brownian motion(e.g., [41]), we can see that:

$$|B_t| \leq Ct^{\frac{1}{2}+\frac{1}{100}} \tag{18}$$

for some absolute constant C, $\mathbb{P}$ - almost surely. Also, basic properties of Brownian motion imply that $\alpha_t = \alpha_1$ in distribution for all t. Moreover, the application of basic probabilistic principles implies that α_1 takes two different real values both with strictly positive probability and hence is not $\mathbb{P}$ - almost surely constant. This immediately implies that:

$$\mathbb{V}ar(\alpha_t) > 0 \tag{19}$$

for all t. The fact that $\Delta_{AB}(t)$ converges to 0 exponentially fast for all $M \in (0, \frac{1}{4\alpha^*})$ coupled with the above tells us that:

$$\mathbb{V}ar(\alpha_t) > \gamma^2t^2\mathbb{V}ar(\alpha_1) - Ct^{\frac{3}{2}+\frac{1}{100}} \tag{20}$$

Using the same logic we can deduce that

$$|A_t| \leq Ct \tag{21}$$

for some absolute constant C, $\mathbb{P}$ - almost surely. This implies that both numerator and denominator in the above tend to 1 as t tends to infinity and hence the proof is complete. We note that the same results hold in case of an arbitrary lag τ with a different constant in the inequality (21) which is equally finite and has no impact on the asymptotic results. ■

Note that, in classical Mathematical Physics, the rigorous definition of a phase transition requires that the correlation between particle spins does not decay with their topological distance in the network. However, this is where the similarity with concepts from Mathematical Physics ends as mathematical methods used in the proofs are completely disjoint from those used in classical derivations, e.g. there is no direct analogy to neither Hamiltonian nor Boltzmann distribution, rather, purely mathematical arguments based on general probabilistic reasoning and usage of well known properties of Brownian motion suffice. In other words, the existence of phase transition does not require the material to become a perfect magnet with perfectly aligned spins. We note the analog of this holds in the case of phase transitions established in the previous result. More specifically, we shall demonstrate that both, with probability one, $\alpha(t) = \alpha_1$ and $\alpha(t) = 1 - \alpha_1$ infinitely often. This means that the switching will never stop with probability one, or in other words, still there is no finite time T (no matter how far in the future) such that from that moment onward $\alpha(t) = \alpha_1$ or $\alpha(t) = 1 - \alpha_1$ for all $t \geq T$. Thus, although the expectation of agents will be perfectly correlated in the long run, agents will still, from time to time, put more weight on either bounded rational or herding component in the same way as observed in Mathematical Physics that phase transition will occur but this does not imply the material will become a magnet with all particles perfectly aligned in their +1 or -1 direction. We will formally prove this result below upon introducing necessary mathematical machinery.

Definition 2: Let $(\Omega, \mathcal{A}, \mathbb{P})$ be some probability space. We say that the event A_n occurs infinitely often if and only if A_n holds for infinitely many n. More formally:

$$\{A_n \ i.o.\} := \{\omega \in \Omega : A_n \text{ occurs for infinitely many } n\} := \limsup_{n\to\infty} := \bigcap_{i=o}^{\infty} \bigcup_{n=i}^{\infty} A_n$$

Theorem 3: For any sequence of events A_n the following relation holds true:

$$\mathbb{P}(\limsup A_n) \geq \limsup \mathbb{P}(A_n)$$

Proof: See Exercise 6.6 on page 49 of [50] ■

Definition 3: Suppose B is a Brownian motion defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define the sigma fields $\mathcal{F}_t$ and $\mathcal{T}$ via:

$\mathcal{F}_t := \sigma(B_s : s \geq t)$ which therefore corresponds to the future at time t. Moreover let $\mathcal{T} := \bigcap_{t\geq 0} \mathcal{F}_t$ stand for the associated tail σ- field.

Theorem 4: If $A \in \mathcal{T}$ then either $\mathbb{P}_x(A) = 0$ or $\mathbb{P}_x(A) = 1$, where $\mathbb{P}_x$ stands for the probability measure associated with the initial condition when Brownian motion starts at x

Proof: See Theorem 2.9 on page 383 from ([50]) ■

Theorem 5: The stochastic process $\alpha(t)$ is infinitely oscillating with probability one. More rigorously we have:

$$\mathbb{P}(\alpha(t) = \alpha_1 \ i.o.) = \mathbb{P}(\alpha(t) = 1 - \alpha_1 \ i.o.) = 1$$

Proof: Let us define the following sets:

$$\begin{aligned} A^t &= \{B(t) \geq \sqrt{t}\} \\ D^t &= \{B(t) \leq -2\sqrt{t}\} \end{aligned} \tag{22}$$

Then:

$$\mathbb{P}(\alpha(t) = \alpha_1 \ i.o) \geq \mathbb{P}(A^t \cap D^t \ i.o) = \mathbb{P}(B(t) \geq \sqrt{t} \ i.o.) \tag{23}$$

But then:

$$\begin{aligned} \mathbb{P}(B(t) \geq \sqrt{t} \ i.o) &= \mathbb{P}(\limsup_{t\to\infty}\{\frac{B_t}{\sqrt{t}} \geq 1\}) \\ &\geq \limsup_{t\to\infty} \mathbb{P}(\frac{B_t}{\sqrt{t}} \geq 1) \\ &= \limsup \mathbb{P}(B_1 \geq 1) > 0 \end{aligned} \tag{24}$$

where we have used both Theorem 3 and the scaling law for Brownian motion namely that:

$$\frac{B_t}{\sqrt{t}} \overset{d}{=} B_1 \overset{d}{=} \mathcal{N}(0, 1) \tag{25}$$

Thus, by Theorem 4 we have:

$$\begin{aligned} \mathbb{P}(B(t) \geq \sqrt{t} \ i.o) > 0 &\implies \mathbb{P}(B(t) \geq \sqrt{t} \ i.o) = 1 \\ &\implies \mathbb{P}(\alpha(t) = \alpha_1 \ i.o.) = 1 \end{aligned} \tag{26}$$

We note that the remaining statement can be established in an identical fashion and thus the proof is complete. ■

Theorem 6: Let $q = 4\alpha^* M$. The rate of convergence of $\mathbb{C}orr(A_t, B_t)$ to unity is exponentially fast, more specifically, there is an absolute constant C depending on model parameters only such that:

$$|\mathbb{C}orr(A_t, B_t) - 1| \leq \frac{Cq^t}{t}$$

for all time periods t simultaneously.

Proof: Using (26), (17), (18), (20) and (21) it can be seen that:

$$\mathbb{C}orr(A_t, B_t) = \frac{1 + \omega_t}{(1 + \epsilon_t)^{\frac{1}{2}}} \tag{27}$$

where ω_t and ϵ_t are functions of time satisfying $\omega_t = \mathcal{O}(\frac{q^t}{t})$ and $\epsilon_t = \mathcal{O}(\frac{q^t}{t})$. Since ϵ_t converges to 0 the sequence $\theta_t := (1+\epsilon_t)^{\frac{1}{2}}$ is uniformly bounded. Moreover,

$$(1+\epsilon_t)^{\frac{1}{2}} = 1 + \frac{\epsilon_t}{2} + \mathcal{O}(\epsilon_t^2) \tag{28}$$

This then together with repeated usage of triangle inequality yields:

$$1 - \mathbb{C}orr(A_t, B_t) = \mathcal{O}(\frac{q^t}{t}) \tag{29}$$

and the proof is complete. ■

Using the same principles, but under somewhat tighter restriction on M, we can prove the analogous result in the case of a two-dimensional regular network with a classical wrap-around designed to ensure all agents have exactly 12 neighbors.

Theorem 7: Let $N \geq 13$ be the number of agents on the two dimensional lattice with the classical wrap-around making sure all agents have exactly 12 neighbors. Assume the same exact conditions as in the previous result.

Then, for all $M \in (0, \frac{1}{24\alpha^*})$

$\mathbb{C}orr(\epsilon_t^i, \epsilon_t^j) \to 1$ as $t \to \infty$

Proof: Observe that the graph $\mathbb{G}$ corresponding to the matrix L is shown in Fig. 2.

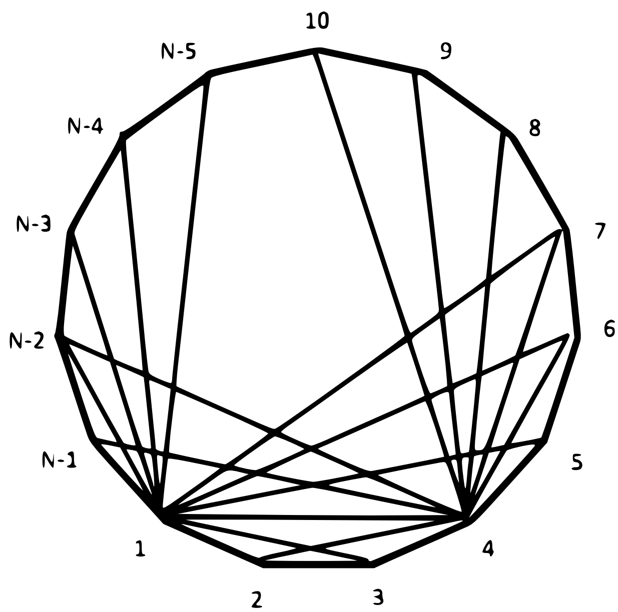

FIGURE 2. Connectivity in a 2D Graph.

The proof of the theorem follows the demonstration for the Theorem 2 given that $\mu(\mathbb{G}) \leq 24$. ■

Moreover, there the convergence of $\mathbb{C}orr(A_t, B_t)$ to unity is again exponentially fast with technical detailed specified as follows:

Theorem 8: Let $q = 24\alpha^* M$. The rate of convergence of $\mathbb{C}orr(A_t, B_t)$ to unity is exponentially fast, more specifically, there is an absolute constant C depending on model parameters only such that:

$|\mathbb{C}orr(A_t, B_t) - 1| \leq \frac{Cq^t}{t}$
for all time periods t simultaneously.

Proof: The proof is identical to the one of the proof of Theorem 6 and is thus neglected. ■

B. NUMERICAL RESULTS

In order to further validate the sharpness of the analytic results derived, we have initiated the network of 500×500 interacting agents by assigning their expectations as independent, uniformly distributed random variables on the interval $[0, 10000]$ and ran it over 512 time periods. By the very construction, the system therefore starts its evolution from the state where the expectations between any pair of agents are uncorrelated. The study of time-evolution over five hundred periods of every pair of agents in this network is computationally infeasible as it would require at least $\binom{250000}{2} \times 512 = \mathcal{O}(10^{13})$ operations. Instead, we use a proxy by sampling 17 pairs of agents which are direct neighbours of one another, 17 pairs of agents which are on a distance 100 from one another and 17 pairs of those which are on the distance of a 1000 from one another, where the distance between agents a_{ij} and a_{lk} is defined as $500(i-l)+j-k$, $i, j, k, l \in \{1 \dots 499\}$, and ij, lk are coordinates of agents in the 2D case. In the 1D case the distance between agent i and agent j is defined as $j-i$ for $j > i$. The rationale behind this approach was to examine whether the correlations between agents are unaffected by their topological distance which is a signature of phase transitions as defined in [15]. Subsequently, for each pair of agents initially sampled, we run the network 60 times in order to obtain an estimate of their correlation at every moment in time based on the maximum likelihood estimator. We note that concept of correlation at every moment in time used above is not be confused with the concept of correlation functions from Mathematical Physics, indeed, they are to be understood in the classical probabilistic sense with the emphasis on the fact they are computed at different points in time which makes them, formally mathematically speaking, functions of time themselves.

Simulations show that the largest values of M for which the phase transition to a perfectly correlated state occurs are $M = 0.28$ and $M = 0.225$ for the case of the networks where each agent has 2 and 12 neighbors, respectively. These values were obtained by means of applying the greedy search based on the bisection method, starting from the theoretical bounds provided by closed form results. In all cases we set $\alpha_1 = 0.9$ for concreteness, noting there are no fundamental changes in results for other values of α_1. We say that the correlations have converged if there is a time point $t_0 < 512$ after which correlation between each sampled pair of agents is equal to 1 for all $t_0 \leq 512$. Typical graph of

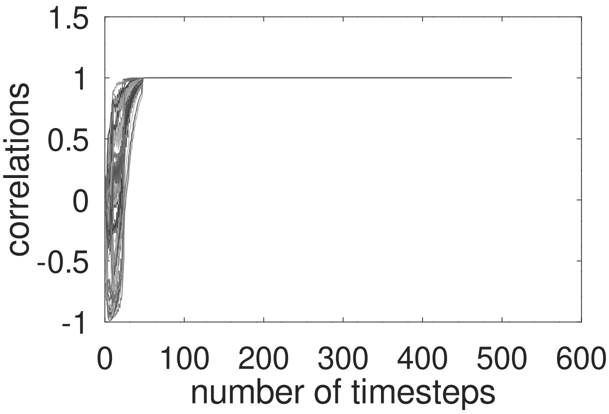

FIGURE 3. **Typical simulated time series of correlations between randomly chosen agents for the 1D case for M smaller than the critical value 0.28. Correlations converge to one after approximately 40 iterations. Different lines correspond to correlation trajectories between randomly selected pairs of agents (51 overall).**

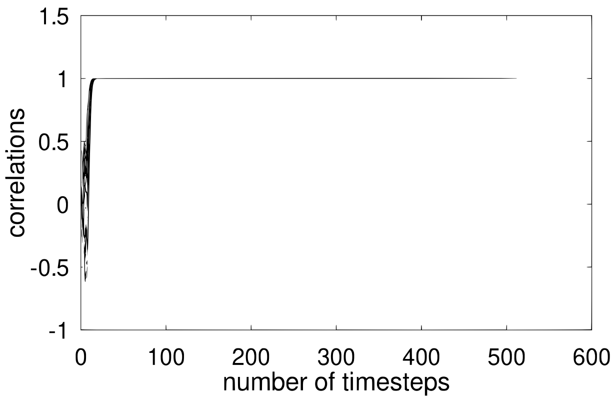

FIGURE 4. **Typical simulated time series of correlations between randomly chosen agents for the 2D case for M_{ij} smaller than the critical value 0.225. Correlations converge to one after approximately 20 iterations. Notation as in Fig. 3.**

converging correlations in time of randomly sampled agents is shown in Fig. 3 and Fig. 4 and corresponds to 1D and 2D cases, respectively. For larger values of M this convergence does not take place with correlations exhibiting erratic trend oscillating in the interval $[-1, +1]$ with typical graph in this case represented in Fig. 5. We note that the theoretical result proven in Theorem 2 is therefore optimal since the largest value of M which results in perfect correlation in the long run reads $M = \frac{5}{18} = 0.2777$ which is an almost perfect match for the computational analysis. In case of the network where each agent has precisely 12 neighbors, however, the largest value of the M for which Theorem 6 is valid reads $M = \frac{5}{108} = 0.0463$ which is smaller than the value obtained computationally ($M = 0.225$). In case of the network where each agent has exactly two neighbours, the corresponding graph is both regular and bipartite, and hence the upper bound on M in Theorem 1 is optimal, following the conditions of the Theorem 1 in [49], while in the other case, where each agent has 12 neighbors, the corresponding graph is,

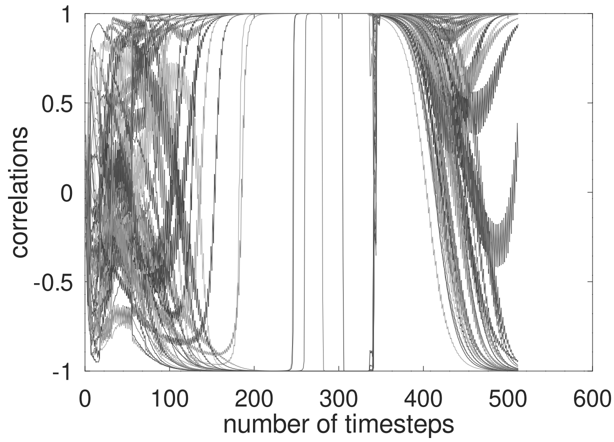

FIGURE 5. **Typical simulated time series of correlations between randomly chosen agents for the 2D case for M larger than the critical value 0.225. Correlations show no convergence trend. Notation as in Fig. 3.**

although regular, not bipartite. We conjecture, that this fact leads to discrepancy in estimates of the upper bound on M obtained theoretically and computationally, noting that establishing this rigorously is beyond the scope of this paper, as it would require sharper results in linear algebra to be proven first. On the purely technical side, we note that one may have to use larger networks in simulations in order to obtain robust estimates of the upper bound on M for which the phase transitions still occur, such as 1000×1000 and beyond. Moreover, we note that convergence to the state of perfect correlation is rapid for both network topologies with A and B iterations needed for the case of the network where each agent has 2 and 12 neighbors, respectively. This is completely in line with the analytic results (Theorem 6 and Theorem 8) which demonstrate that the convergence to the state of perfect correlation is indeed exponentially fast. We have performed additional simulations to study the behavior of the system when the interactions between agents are not necessarily symmetric. To be more specific, for fixed value of M, we assumed that M_{ij} s are independent uniformly distributed random variables on the interval $[0, M]$. We find that asymmetry has little to no effect on the general behavior of the system, the only difference we observed is in the largest value of M which leads to perfect correlation between agents' expectations in the long run, the corresponding value for 1D network reads $M = 0.57$ while the corresponding counterpart for the 2D network reads $M = 0.13$. Finally, we have investigated the effect of introducing asynchronous updating by adding an additional individual-agent level of randomness where a specified percentage of agents ignores the triggering event (change in the weights of bounded rational and herding component). The perturbed system no longer exhibits perfect correlation in the long run, however, it does still remains consistent with the definition of phase transitions in the classical sense of Onsager [17]. To be more specific, we find that, in 1D, the minimal correlation observed reads 0.16, the maximal correlation reads 0.95 while the average correlation reads 0.6. Similarly, in the 2D case, we find that the minimal correlation reads 0.11, maximal correlation reads 0.92 while the average correlation reads 0.61.

IV. SUMMARY AND CONCLUSION

Investigators in engineering and related applications have recognized correspondences between dynamics of financial markets and information theory (e.g., [33], [51], [52]). Investigators continue to cite a correspondence of the models they propose to ferromagnetic 2D Ising models (e.g., [7], [8]). These have primarily been in applications to binary states in opinion dynamics of networked agents (e.g., [4]) and price in financial markets (e.g., [7]). These authors have shown that their models can demonstrate phase transitions between ordered and disordered states of agents in the models. These demonstrations have been in numerical exercises.

We address on processing in expectations for price in financial markets and propose a multicomponent model of the dynamics that generate price expectations. The multicomponent model of interacting agents we posit represents components in bounded rational behavior and copying behavior that generates herding in financial markets. A transition function defines the weight of the components in agent expectations.

In the analytic application to dynamics of price expectations, we propose a first component of the model that represents price in financial markets as Brownian motion with drift. The second component represents the influences of other agents. Following Onsager's methodology in a 2D Ising model ([17]), we analytically show that the stochastic model can generate phase transitions that have been the most cited property of a 2D Ising model. More formally, the system is said to exhibit a phase transition if and only if there exists a strictly positive real number ϵ such that for all pairs of agents in the network the correlation between their spins is larger than or equal to ϵ no matter how far apart they are.

The modeling choices made in this manuscript are largely consistent with those typically undertaken in the literature as described in the paper ([53]). The authors provide a detailed breakdown of the statistics of different modeling techniques/options used by researchers in agent-based models of opinion forming, computed over more than 150 papers on the corresponding subject. The opinion itself can be modeled as either a discrete (18.5 % of papers) or a continuous one, where the later can be further split between continuous over bounded interval (45.7 % of papers) or continuous over the entire real line (35.8 % of papers). We note that the modeling choice used in this manuscript corresponds to the continuous expectations which is in line with the majority of the research performed. Moreover, we note that the expectations are initialized as independent and uniformly distributed random variables on the interval [0,10000] while its time evolution is otherwise unrestricted and can in principle take any real value which therefore represents a hybrid of approaches undertaken by other researchers.

The interaction dynamics can be bilateral (88 % of papers) while symmetric models correspond to only 18 % of papers. The modeling choice used in this paper is indeed bilateral while symmetry is also assumed, however, we have equally investigated the effects of lack of symmetry on the system's behavior. Agents can either interact pairwise (35.8 % of papers), to closest neighbors (35.1 % of papers) or any-to-any (29.1 % of papers). The modeling choice taken in this paper corresponds to closest neighbors which is the second most popular modeling option chosen by the other researchers.

The updating function can be either non-linear (79.5 % of papers) or linear (20.5 % of papers) and thus the modeling choice is in line with the one taken by most of the researchers. Finally, the updating frequency can be either periodic (when all agents update opinions in all time steps) or aperiodic if this assumption is violated in some fashion. Specific possibilities could be that only a couple of agents change their opinions at each time step (and it is not known in advance when their turn comes again), that opinions are updated once a triggering effect takes place or when a random selection/proportion of agents are changing their opinion. The modelling choice undertaken in this manuscript is a hybrid of the ones used by other researchers as the major updating indeed corresponds to the stochastic triggering effect, but this effect was initially assumed to be the same for all agents. However, we have also investigated the effects of breaking the synchronicity in the updates by introducing the extra level of randomness on the level of individual agents as elaborated in the results.

The analytical results we report are based on the definition of phase transitions as introduced in [17]. More specifically, in Onsager's precise definition of a phase transition, it occurs when a thermodynamic quantity (like magnetization, specific heat, or susceptibility) changes its behavior non-analytically at a certain temperature. For the 2D Ising model, Onsager showed that while the free energy is continuous across T_c, its second derivative (specific heat) diverges logarithmically at T_c. This was the first mathematically rigorous demonstration of a continuous (second-order) phase transition. Thus, phase transitions were no longer just phenomenologically (i.e. observed experimentally) but defined mathematically in terms of singularities in thermodynamic functions. Whereas Onsager's demonstration shows that the equivalent of agents tends to a positive correlation in signs, we demonstrate conditions under which agents will tend to perfect correlation. We note this phenomena corresponds to the case of synchronicity between agents that others have identified presaging critical points in market prices (e.g., [23]). The results we report in both 1D and 2D directly support numerical demonstrations of phase transitions in financial markets that have been related to critical points in price dynamics. In case of the one-dimensional lattice, theoretical results are indeed best possible as corroborated by the corresponding computer simulations whereas they can in theory be improved in the case of the two-dimensional lattice, however, this would require further breakthrough in mathematical research of the associated problems in linear algebra. Taken together, our analytical results support and extend theory and evidence on phase transitions in multicomponent models of expectations for price in financial markets.

Finally, we note that the demonstration of phase transitions we provide for a multicomponent model of price expectation in financial markets can be extended to a range of other models of opinion dynamics and information processing (e.g., [23]) in which an aggregate of micro-level interactive processes between networked agents exhibits a good asymptotic approximation by Brownian motion. Moreover, there is number of additional noteworthy contributions in this direction which we will briefly summarize below. Zubillaga et al. ([54]) introduces a three-state agent-based model where agents can be either buying, selling or staying inactive. For certain parameter ranges the system exhibits an ordered phase with the presence of large clusters of agents that share the same state, which results in spontaneous magnetization. They equally demonstrate that their model can reproduce qualitative and quantitative features of the real financial time series, including distribution of returns with long tails, volatility with long term memory and volatility clustering. The price is identified with the magnitude of magnetization defined by an analogy to the 3-state Potts model. Lux ([55]) proposes an agent-based model where each agent has 2 possible opinions, optimistic and pessimistic. These evolve/change in continuous time with a Poisson process formalizing the switch between them. In order to estimate the parameters of such models the author uses a numerical maximum likelihood procedure in higher dimensions. This work equally demonstrates there is no necessarily a conflict between agent-based models and models of aggregate data via structural equations. Gontis et al. ([56]) use a version of the 3-state agent-based herding model to describe the endogenous dynamics of agents in the financial markets and to reproduce the statistical properties of volatility return intervals. They derive macroscopic equations based on the microscopic herding interactions. Zheng et al. ([57]) introduce an asset pricing model with boundedly rational agents who are specified as using simple heuristics in their decision making. These authors study the impact of the investor's behavior on the return volatility of the risky asset and find that the speed of the adjustment of a market maker can be a source of the volatility persistence, thus a potential source of the long memory. Wang et al. ([58]) use a 2D 3-state Potts model on a square lattice to produce a financial agent-based price model. The 3 states equally correspond to buying, selling and making no trading decision. Empirical studies indicate that the proposed return intervals from the Potts model and real stock market indices hold similar statistical properties. More specifically, the authors use a multi-scale multifractal detrended cross correlation analysis and Lempel-Ziv complexity to perform numerical research on the return intervals in China's stock indices. Other important agent-based opinion dynamics models in social sciences include the work of Sznajd-Weron and Sznajd ([59]) who introduced a simple Ising model which can describe a mechanism of making a decision in a closed community. The rules describe the influence of a given pair on the direction of its nearest neighbors. The random variable needed for an agent to change opinion is shown to have a power-law distribution. They demonstrate that in the long run, within the closed community, there are only 2 possibilities, either dictatorship or a stalemate where no decision can be made. Azhari et al. ([60]) generalize the Sznajd model to the case when there are r influencers and n agents to be persuaded. They investigate their model on a 2D square lattice with different influencer configurations. They demonstrate that the system exhibits continuous phase transition when $r \leq 5$ and discontinuous phase transition once $r > 5$, for all possible values of n. The number of persuaded agents n only impacts the time needed to reach the equilibrium, t_{steady} is proportional to $\frac{1}{n}$. Agents in the model exhibit two types of behavior, conformity and independence, conformists follow the opinion of the majority while the independent agents change their opinions without being influenced by the group's opinion. Finally, Bizyaeva et al. ([61]) propose a general nonlinear opinion dynamics model and demonstrate that it exhibits a rich variety of opinion formation behaviors governed by bifurcations. These authors established the existence of agreement and disagreement equilibria and multi-stability of opinion formation outcomes arise from bifurcation of the general opinion dynamic model in a mathematically rigorous fashion.

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