



The system of translates and the special affine Fourier transform

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Abstract

The translation operator T^A associated with the special affine Fourier transform (SAFT) \mathcal{F}_A is introduced from harmonic analysis point of view. The analogues of Wendel's theorem, Wiener theorem, Wiener-Tauberian theorem and Bernstein type inequality in the context of the SAFT are established. The shift invariant space V_A associated with the special affine Fourier transform is introduced and studied along with sampling problems.

Keywords Convolution · Poisson summation formula · Shift invariant space · Wendel's theorem · Zak transform

Mathematics Subject Classification Primary 42A38 · Secondary 42A85 · 42C15

1 Introduction and background

The special affine Fourier transform (SAFT) was first considered by S. Abe and J. T. Sheridan in [1] for the study of certain operations on optical wave functions. The SAFT is formally defined as

$$\mathcal{F}_A f(\omega) = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(t) e^{\frac{i}{2b}(at^2 + 2pt - 2\omega t + d\omega^2 + 2(bq - dp)\omega)} dt, \quad \omega \in \mathbb{R}, \quad (1.1)$$

where A stands for the set $\{a, b, c, d, p, q\}$ of real parameters which satisfy the relation $ad - bc = 1$, $b \neq 0$. The integral transform (1.1) is related to the special affine linear transform of the phase space

$$\begin{pmatrix} t' \\ \omega' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ \omega \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix}. \quad (1.2)$$

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Due to the conditions on the parameters a, b, c, d the matrix in (1.2) belongs to the special linear group $SL(2, \mathbb{R})$ and the affine transform (1.2) is therefore given by elements from the inhomogeneous linear group

$$ISL(2, \mathbb{R}) = \left\{ \begin{pmatrix} M & v \\ 0 & 1 \end{pmatrix} : M \in SL(2, \mathbb{R}), v \in \mathbb{R}^2 \right\}.$$

This justifies the name special affine Fourier transform for (1.1). The action of the group $SL(2, \mathbb{R})$ on the time frequency plane and the relation to quadratic Fourier transforms is well studied. We will not go into the details here but refer to the book [13].

A number of important transforms are special cases of the SAFT. For example, $A = \{0, 1, -1, 0, 0, 0\}$ gives the ordinary Fourier transform and $A = \{0, -1, 1, 0, 0, 0\}$ its inverse. The parameter set $A = \{\cos \theta, \sin \theta, -\sin \theta, \cos \theta, 0, 0\}$ gives the fractional Fourier transform, and $A = \{1, \lambda, 0, 1, 0, 0\}$ produces the Fresnel transform.

In optics, certain one parameter subgroups of the $ISL(2, \mathbb{R})$ are of special interest. Among them are the fractional Fourier transform, the Fresnel transform (also called free space propagation in this context), the hyperbolic transform $A = \{\cosh \theta, \sinh \theta, \sinh \theta, \cosh \theta, 0, 0\}$, the lens transform $A = \{1, 0, \lambda, 1, 0, 0\}$, and the magnification transform $A = \{e^\beta, 0, 0, e^{-\beta}, 0, 0\}$. The latter two cases need a careful analysis for the limit case $b \rightarrow 0$ which we will not consider in this paper. We will not try to expound the various connections to optics further but refer to [17] for more details.

In this paper, we consider (1.1) from the point of view of applied harmonic analysis and take it as a signal transform of a (suitable) function. We are mainly interested in studying the principal shift invariant spaces and sampling theorems related to the SAFT. In the classical case, the principal shift invariant space generated by $\phi \in L^2(\mathbb{R})$ is defined as $V(\phi) = \overline{\text{span}}\{T_k \phi = \phi(\cdot - k) : k \in \mathbb{Z}\}$. The classical Fourier transform ($A = \{0, 1, -1, 0, 0, 0\}$ in (1.1)) plays a crucial role for the analysis of such spaces. The crucial point is that the ordinary translation and the classical Fourier transform are intimately related. This is due to the identity $e^{i\omega(t-x)} = e^{i\omega t} e^{-i\omega x}$ which gives, as a consequence the convolution theorem, the relation between translation, modulation, Fourier transform etc. These theorems are used over and over again in Fourier analysis and in the study of shift invariant spaces in particular. It is completely obvious that the ordinary translation does not interact nicely with the SAFT (resp. its complex exponential kernel). Hence working with the SAFT, a new concept of a translation is needed. This generalized translation should be linked to the SAFT in an analogous manner as the ordinary translation is linked to the classical Fourier transform. If this is the case, then it seems reasonable to expect that the central theorems (convolution theorem etc.) hold in a similar manner. An idea for the construction of such generalized translation comes from the observation that in the classical setting we have

$$T_x f(t) = \mathcal{F}^{-1}(e^{-i\omega x} \mathcal{F} f)(t).$$

In this paper, we define a new translation operator T_x^A , which serves our purpose in the case of SAFT.

Considering the SAFT as a general integral transform and using the notion of the generalized translation operator and convolution as in [19], the authors in [31] introduced the concepts of the translation operator and convolution in connection with the SAFT. However, the concept of the translation operator is not very explicit in this setup. In [29], the authors introduced the notion of convolution using chirp modulation operator and obtained a convolution theorem in the context of the SAFT. In this paper, we introduce A -translation operator in connection with the SAFT and obtain a convolution theorem using it. Here, we point out that our notion of convolution and the one in [29] are the same.

Due to the lack of an explicit generalized translation operator, the consequences of this concept with respect to harmonic analysis were not explored. In this paper, we shall demonstrate that the generalized translation is the suitable concept to obtain analogues of fundamental theorems such as Wendel's theorem for multipliers, Wiener theorem and Wiener-Tauberian theorem in connection with the closed ideals of translation invariant spaces in the context of the SAFT. For a study of multipliers and Wendel's theorem for the Fourier transform we refer to [21], for Wiener theorem and Wiener-Tauberian theorem we refer to [14] and [25]. The novelty of this approach is that apart from these theorems, one can look into the study of multiplier theory, including Hörmander multiplier theorem in the SAFT domain. Moreover, one can define an appropriate modulation operator in connection with the SAFT, which can be used to define corresponding new modulation spaces. This in turn, motivates to study multiplier results for the new modulation spaces. (See [9] in this connection.)

Using the new translation operator T_x^A the A -shift invariant spaces are defined as $V_A(\phi) = \overline{\text{span}}\{T_k^A \phi : k \in \mathbb{Z}\}$ for an appropriate function $\phi \in L^2(\mathbb{R})$. When ϕ belongs to the Wiener amalgam space $W(C_0, \ell^1)(\mathbb{R})$ the space $V_A(\phi)$ turns out to be a reproducing kernel Hilbert space. In the theory of shift-invariant spaces, it is interesting to obtain necessary and sufficient conditions on ϕ such that the system of translates $\{T_k \phi : k \in \mathbb{Z}\}$ forms a frame sequence, Riesz sequence or an orthonormal system. For a proof of these characterizations, see Theorem 9.2.5 in [12]. In this paper, we give characterization theorems for the system of A -translates $\{T_k^A \phi : k \in \mathbb{Z}\}$ to be a frame sequence, orthonormal system or a Riesz sequence. If the system of translates is a frame, then an important question is about the nature of the dual frame elements. We show that in our setting, the elements of the dual frame of system of A -translates are also A -translates of a single function.

In the final part of the paper we study the sampling in A -shift invariant spaces. A fundamental problem in sampling theory is to find, for a certain class of functions, appropriate conditions on a countable sampling set $X = \{x_j \in \mathbb{R} : j \in J\}$ under which a given function $f \in V$ can be reconstructed uniquely and stably from the samples $\{f(x_j) : j \in J\}$. We refer to the work of Butzer and Stens [10] for a review on sampling theory and its history. When V is the classical principal shift invariant space with a single generator $V(\phi)$ or multi-generators $V(\phi_1, \phi_2, \dots, \phi_r)$, $\phi, \phi_1, \phi_2, \dots, \phi_r \in L^2(\mathbb{R})$, there is a huge literature available on several interesting problems connected with sampling theory starting from the fundamental Shannon sampling theorem. We cite only a few references in this connection for the reader to get familiarity with this subject matter. (See [2–5, 11, 15, 16, 20, 22, 23, 27, 28, 30, 32]). In [6, 7] Bhandari and Zayed have defined chirp-modulated shift invariant spaces in connection with fractional Fourier transform, SAFT respectively and studied sampling problems in it.

In this paper, similar to Theorem 4.2 of [3] and Theorem 2.1 of [26], we obtain equivalent conditions for a set $X = \{x_j : j \in \mathbb{Z}\}$ to be a stable set of sampling for $V_A(\phi)$ in terms of the operator U where $U_{jk} = T_k^A \phi(x_j)$, the reproducing kernel and the Zak transform Z_A , which we introduce here. We also obtain a sufficient condition for the set of integers to be a stable set of sampling for $V_A(\phi)$. In the study of non uniform sampling and average sampling, Bernstein type inequalities play an important role. In this paper we obtain an analogue of Bernstein type inequality for $V_A(\phi)$. However, we do not intend to study non-uniform sampling and average sampling in this paper. We focus on uniform sampling. In particular, when \mathbb{Z} turns out to be a stable set of sampling, we obtain a reconstruction formula and hence a sampling theorem in the sense of L^2 convergence for certain A -shift invariant spaces $V_A(\phi)$. Further, under some additional hypotheses on ϕ , we obtain a sampling formula in the sense of L^2 convergence, pointwise convergence and uniform convergence. As corollaries we obtain Shannon sampling theorem in the SAFT domain and sampling theorem for the A -shift invariant space generated by second order B -spline.

We organize our paper as follows. In Sect. 2, we define A -convolution of a measure and a function and chirp modulation $C_s\mu$ of a measure μ . Using this and the function $\rho_A(x) = e^{\frac{i}{2b}(ax^2+2px)}$, we obtain a relation between classical translation and A -translation. We prove an analogue of Wendel's theorem for the SAFT. In Sect. 3, we study closed ideals in the Banach algebra $(L^1(\mathbb{R}), \star_A)$. We obtain analogues of Wiener theorem and Wiener-Tauberian theorem in the context of the SAFT. In Sect. 4, we study A -shift invariant spaces and their theoretical aspects. Section 5 as well as Sect. 6 are devoted to sampling theorems in A -shift invariant spaces. Finally in Sect. 7, we present a local reconstruction method for sampling in A -shift invariant spaces.

Now we shall provide the necessary terminology and background for this paper.

Let $0 \neq \mathcal{H}$ be a separable Hilbert space.

Definition 1.1 A sequence $\{f_k : k \in \mathbb{N}\}$ of elements in \mathcal{H} is a *frame* for \mathcal{H} if there exist $m, M > 0$ such that

$$m\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq M\|f\|^2, \quad f \in \mathcal{H}.$$

The numbers m, M are called *frame bounds*. If we have the right hand side inequality for a sequence in \mathcal{H} , then that sequence is called a *Bessel sequence*.

Definition 1.2 Let $\{f_k : k \in \mathbb{N}\}$ be a Bessel sequence in \mathcal{H} , then the *synthesis operator* $T : \ell^2 \rightarrow \mathcal{H}$ is defined by

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \quad \{c_k\} \in \ell^2.$$

The adjoint of T is given by $T^*(f) = \{\langle f, f_k \rangle\}$, $f \in \mathcal{H}$, called the *analysis operator*. Composing T and T^* we obtain the *frame operator*

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

The operator S is invertible. Further if $\{f_k : k \in \mathbb{N}\}$ is a frame for \mathcal{H} , then $\{S^{-1}f_k : k \in \mathbb{N}\}$ is also a frame for \mathcal{H} and it is called the *canonical dual frame* of the frame $\{f_k : k \in \mathbb{N}\}$.

Definition 1.3 A sequence $\{f_k : k \in \mathbb{N}\}$ in \mathcal{H} is said to be a *Riesz basis* if there exist a bounded invertible operator T on \mathcal{H} and an orthonormal basis $\{u_k : k \in \mathbb{N}\}$ of \mathcal{H} such that $f_k = Tu_k$, $\forall k \in \mathbb{N}$. The sequence $\{f_k : k \in \mathbb{N}\}$ is called a *Riesz sequence* if it is a Riesz basis for its closed linear span.

Equivalently $\{f_k : k \in \mathbb{N}\}$ is a *Riesz sequence* if there exist $m, M > 0$ such that

$$m\|c\|^2 \leq \left\| \sum_{k=1}^{\infty} c_k f_k \right\|^2 \leq M\|c\|^2, \quad \text{for every finite sequence } \{c_k\}.$$

Definition 1.4 Let $\{f_k : k \in \mathbb{N}\}$ be a Riesz basis for \mathcal{H} . The *dual Riesz basis* of $\{f_k : k \in \mathbb{N}\}$ is the unique sequence $\{g_k : k \in \mathbb{N}\}$ in \mathcal{H} satisfying

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

The dual Riesz basis is often called as the *biorthogonal Riesz basis*.

Definition 1.5 The *Gramian* associated with the Bessel sequence $\{f_k : k \in \mathbb{N}\}$ is an operator on ℓ^2 whose jk^{th} entry in the matrix representation with respect to the canonical orthonormal basis is $\langle f_k, f_j \rangle$.

It is well known that a sequence $\{f_k : k \in \mathbb{N}\}$ is a Riesz sequence if there exist $m, M > 0$ such that its Gramian G satisfies the following inequalities:

$$m\|c\|^2 \leq \sum_{k \in \mathbb{N}} |\langle Gc_k, c_k \rangle|^2 \leq M\|c\|^2, \quad \forall c = \{c_k\} \in \ell^2.$$

Definition 1.6 A closed subspace V in $L^2(\mathbb{R})$ is said to be a *shift invariant space* if $f \in V \Rightarrow T_k f \in V$, $\forall k \in \mathbb{Z}$, $f \in V$, where $T_k f(t) = f(t - k)$.

In particular, for $\phi \in L^2(\mathbb{R})$, $V(\phi) = \overline{\text{span}}\{T_k \phi : k \in \mathbb{Z}\}$ is called the *principal shift invariant space*.

We refer to [12], for a study of frames, Riesz basis and shift invariant spaces.

Definition 1.7 A set $X = \{x_k \in \mathbb{R} : k \in \mathbb{Z}\}$ is said to be a *stable set of sampling* for a closed subspace V of $L^2(\mathbb{R})$ if there exist constants $0 < m \leq M < \infty$ such that

$$m\|f\|^2 \leq \sum_{k \in \mathbb{Z}} |f(x_k)|^2 \leq M\|f\|^2,$$

for every $f \in V$.

Definition 1.8 The *Wiener amalgam space* $W(C_0, \ell^p)(\mathbb{R})$, $1 \leq p < \infty$ is defined as

$$W(C_0, \ell^p)(\mathbb{R}) = \{f \in C(\mathbb{R}) : \sum_{k \in \mathbb{Z}} \max_{x \in [0, 1]} |f(x + k)|^p < \infty\}.$$

2 The new translation

In this section, we introduce A -translation operator in connection with the SAFT. Using A -translation operator, we define A -convolution of a regular Borel measure and a function. Further, we obtain an analogue of Wendel's theorem. Towards this end, first we extend the definition of SAFT to the space of all regular Borel measures.

Definition 2.1 Let $f \in L^1(\mathbb{R})$. Then the *special affine Fourier transform* is defined as

$$\mathcal{F}_A f(\omega) = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(t) e^{\frac{i}{2b}(at^2 + 2pt - 2t\omega + d\omega^2 + 2(bq - dp)\omega)} dt, \quad \omega \in \mathbb{R}, \quad (2.1)$$

where A stands for the set of six parameters $\{a, b, c, d, p, q\} \subset \mathbb{R}$ with $ad - bc = 1$ and $b \neq 0$.

With the help of the following auxiliary functions

$$\eta_A(\omega) = e^{\frac{i}{2b}(d\omega^2 + 2(bq - dp)\omega)}, \quad (2.2)$$

$$\rho_A(t) = e^{\frac{i}{2b}(at^2 + 2pt)}, \quad (2.3)$$

the SAFT can be expressed as

$$\mathcal{F}_A f(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (\rho_A f) \wedge (\omega/b), \quad (2.4)$$

where \widehat{f} denotes the Fourier transform of f . Since $|\eta_A(\omega)| = 1 = |\rho_A(t)|$ for all $t, \omega \in \mathbb{R}$, we immediately see from (2.4) that $\mathcal{F}_A f \in C_0(\mathbb{R})$ with $\|\mathcal{F}_A f\|_\infty \leq (2\pi|b|)^{-1/2} \|f\|_1$. Moreover, (2.4) also shows that \mathcal{F}_A can be extended to $L^2(\mathbb{R})$ and defines a unitary operator on that space. In particular,

$$\|\mathcal{F}_A f\|_2 = \|f\|_2.$$

The inverse of \mathcal{F}_A on $L^2(\mathbb{R})$ can also be easily determined using (2.4)

$$\mathcal{F}_A^{-1} f(t) = \frac{\overline{\rho_A}(t)}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} f(\omega) \overline{\eta_A}(\omega) e^{i\omega t/b} d\omega.$$

Finally, (2.4) also provides an extension of the SAFT to $M(\mathbb{R})$, the space of all complex valued bounded regular Borel measures on \mathbb{R} , equipped with the total variation norm. For $\mu \in M(\mathbb{R})$, we have

$$\mathcal{F}_A \mu(\omega) = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} e^{\frac{i}{2b}(at^2 + 2pt - 2t\omega + d\omega^2 + 2(bq - dp)\omega)} d\mu(t), \quad \omega \in \mathbb{R},$$

with $\mathcal{F}_A \mu \in C_b(\mathbb{R})$ and $\|\mathcal{F}_A \mu\|_\infty \leq (2\pi|b|)^{-1/2} \|\mu\|_{M(\mathbb{R})}$.

We now introduce the generalized translation operator associated with SAFT. In order to do so, we fix the following notation

$$T_x f(t) = f(t - x). \quad (2.5)$$

$$M_\omega f(t) = e^{i\omega t} f(t). \quad (2.6)$$

Definition 2.2 Let $x \in \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then A -translation of f by x , denoted by $T_x^A f$, is defined as

$$T_x^A f(t) = T_x M_{-ax/b} f(t) = e^{-i\frac{a}{b}x(t-x)} f(t - x), \quad t \in \mathbb{R}.$$

It is easy to see that T_x^A is norm preserving in all spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ or $C_0(\mathbb{R})$.

We can relate our new translation and the classical translation in the following way, using ρ_A and the chirp modulation operator $C_{a/b}$, where

$$C_s f(t) = e^{i\frac{s}{2}t^2} f(t). \quad (2.7)$$

$$C_{a/b} T_x^A f = e^{i\frac{a}{2b}x^2} T_x(C_{a/b} f) \quad (2.8)$$

$$\rho_A T_x^A f = \rho_A(x) T_x(\rho_A f). \quad (2.9)$$

In fact,

$$C_{a/b} T_x^A f(t) = e^{i\frac{a}{2b}t^2} e^{-i\frac{a}{b}x(t-x)} f(t - x) = e^{i\frac{a}{2b}x^2} e^{i\frac{a}{2b}(t-x)^2} f(t - x) = e^{i\frac{a}{2b}x^2} T_x(C_{a/b} f)(t).$$

Similarly, one can show (2.9).

Now, we collect the properties of T_x^A .

Proposition 2.3 We have the following

- (i) $T_x^A T_y^A = e^{-i\frac{a}{b}xy} T_{x+y}^A, \quad x, y \in \mathbb{R}$.
- (ii) $(T_x^A)^* = e^{-i\frac{a}{b}x^2} T_{-x}^A, \quad x \in \mathbb{R}$.
- (iii) Let $\chi_\omega(t) = \overline{\rho_A}(t) e^{i\omega t/b}$. Then $T_x^A \chi_\omega(t) = \overline{\chi_\omega}(x) \chi_\omega(t)$.
- (iv) Let $f \in L^1(\mathbb{R})$. Then

$$\mathcal{F}_A(T_x^A f)(\omega) = \rho_A(x) e^{-ix\omega/b} \mathcal{F}_A f(\omega), \quad x, \omega \in \mathbb{R}. \quad (2.10)$$

Proof The proof is straightforward. \square

It is interesting to note that the map $x \mapsto T_x$ is a group representation, whereas from Proposition 2.3 (i) it follows that $x \mapsto T_x^A$ is just a projective representation in general, which shows that the new translation is fundamentally different from the classical translation. Proposition 2.3 (iii) is what is known in harmonic analysis as a product formula. The relation (2.10) extends to functions $f \in L^2(\mathbb{R})$ as well. For those functions we have in particular

$$T_x^A f = \mathcal{F}_A^{-1}(\rho_A(x)e^{-ix \cdot /b} \mathcal{F}_A f), \quad (2.11)$$

where the equality holds in the sense of $L^2(\mathbb{R})$ functions.

Definition 2.4 Let $\mu \in M(\mathbb{R})$ and $s \in \mathbb{R}$, then $C_s \mu$ is defined by $d(C_s \mu)(x) = e^{\frac{i}{2}sx^2} d\mu(x)$.

Clearly, $C_s \mu \in M(\mathbb{R})$. Similarly, we can define $\rho_A \mu$ as $d(\rho_A \mu)(x) = \rho_A(x) d\mu(x)$.

Using the A -translation, we define the A -convolution of $\mu \in M(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ as

$$(\mu \star_A f)(t) = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} T_s^A f(t) d\mu(s). \quad (2.12)$$

The integral in (2.12) can also be viewed as a vector-valued integral as follows.

$$\mu \star_A f = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} T_s^A f d\mu(s), \quad (2.13)$$

where the right hand side is a Bochner integral. The convergence of the integral follows from

$$\int_{\mathbb{R}} \|T_s^A f\|_1 d\mu(s) = \|f\|_1 \mu(\mathbb{R}) < \infty.$$

Now we give a relation between classical convolution and A -convolution of a measure and a function. Consider

$$\begin{aligned} (\mu \star_A f)(x) &= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} T_s^A f(x) d\mu(s) \\ &= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} e^{i\frac{a}{2b}s^2} C_{a/b}^{-1} T_s C_{a/b} f(x) d\mu(s) \\ &= \frac{e^{-i\frac{a}{2b}x^2}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} T_s C_{a/b} f(x) e^{i\frac{a}{2b}s^2} d\mu(s) \\ &= \frac{e^{-i\frac{a}{2b}x^2}}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} T_s C_{a/b} f(x) d(C_{a/b}\mu)(s) \\ &= \frac{e^{-i\frac{a}{2b}x^2}}{\sqrt{|b|}} (C_{a/b}\mu \star C_{a/b} f)(x), \end{aligned}$$

using (2.8), which in turn implies that

$$C_{a/b}(\mu \star_A f) = \frac{1}{\sqrt{|b|}} (C_{a/b}\mu \star C_{a/b} f). \quad (2.14)$$

Further, using (2.9), one can show that

$$\rho_A(\mu \star_A f) = \frac{1}{\sqrt{|b|}} (\rho_A \mu) \star (\rho_A f). \quad (2.15)$$

The convolution theorem for the SAFT reads as follows.

Proposition 2.5 Let $\mu \in M(\mathbb{R})$ and $f \in L^1(\mathbb{R})$. Then

$$\mathcal{F}_A(\mu \star_A f)(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A \mu(\omega) \mathcal{F}_A f(\omega). \quad (2.16)$$

Proof The proof is straightforward using the operator ρ_A . \square

In particular, if $\mu = g \, dt$, for some $g \in L^1(\mathbb{R})$ then

$$\mathcal{F}_A(g \star_A f)(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A g(\omega) \mathcal{F}_A f(\omega).$$

See [7] for more details.

The concept of chirp modulation was used in [7] to define a chirp convolution and to get sampling theorems. Although the A -translation is somehow included implicitly in the definition of the chirp convolution, it has not been used to its full extent in [7] and hence the harmonic analysis of the special affine Fourier transform has not been developed. However we want to make use of our new translation from harmonic analysis point of view. Towards this end we first prove an analogue of Wendel's theorem for the SAFT.

Theorem 2.6 (Wendel) Let $T : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ be a bounded linear operator. Then the following statements are equivalent.

- (i) $TT_s^A = T_s^A T$, for all $s \in \mathbb{R}$.
- (ii) $T(f \star_A g) = Tf \star_A g = f \star_A Tg$, for all $f, g \in L^1(\mathbb{R})$.
- (iii) There exists a unique $\mu \in M(\mathbb{R})$ such that $Tf = \mu \star_A f$.
- (iv) There exists a unique $\mu \in M(\mathbb{R})$ such that

$$\mathcal{F}_A(Tf)(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A \mu(\omega) \mathcal{F}_A f(\omega).$$

- (v) There exists a unique $\phi \in L^\infty(\mathbb{R})$ such that $\mathcal{F}_A(Tf)(\omega) = \phi(\omega) \mathcal{F}_A f(\omega)$.

Proof Let $E_{\rho_A} f(t) = \rho_A(t) f(t)$ and define $\tilde{T} : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ by $\tilde{T} = E_{\rho_A} T E_{\overline{\rho_A}}$. Then using (2.9) we get

$$T T_x^A = T E_{\overline{\rho_A}} E_{\rho_A} T_x^A = \rho_A(x) T E_{\overline{\rho_A}} T_x E_{\rho_A} = \rho_A(x) E_{\overline{\rho_A}} \tilde{T} T_x E_{\rho_A},$$

which shows that $T T_x^A = T_x^A T$ iff $\tilde{T} T_x = T_x \tilde{T}$. Similarly we can show that $T(f \star_A g) = Tf \star_A g$ iff $\tilde{T}(f \star g) = \tilde{T} f \star g$, and $Tf = \mu \star_A f$ iff $\tilde{T} f = \frac{1}{\sqrt{|b|}} (E_{\rho_A} \mu) \star f$. The equivalence of (i), (ii), and (iii) now follows from Wendel's theorem in the classical case. That (iii) implies (iv) and (iv) implies (v) is obvious. To show that (i) follows from (v), let $f \in L^1(\mathbb{R})$. For $s \in \mathbb{R}$ we have

$$\begin{aligned} \mathcal{F}_A(T T_s^A f)(\omega) &= \phi(\omega) \mathcal{F}_A(T_s^A f)(\omega) = \phi(\omega) \rho_A(\omega) e^{-i s \omega / b} \mathcal{F}_A f(\omega) \\ &= \rho_A(\omega) e^{-i s \omega / b} \mathcal{F}_A(T f)(\omega) = \mathcal{F}_A(T_s^A T f)(\omega). \end{aligned}$$

\square

We end this section by establishing an analogue of the Poisson summation formula for the SAFT and the corresponding A -translation. From now on, we use the following notation.

$$I := [-|b|\pi, |b|\pi].$$

Theorem 2.7 Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the following formula holds.

$$\sqrt{2\pi|b|} \sum_{k \in \mathbb{Z}} T_{-2kb\pi}^A f(t) e^{-2iabk^2\pi^2} = \sum_{n \in \mathbb{Z}} \overline{\eta_A}(n+p) \mathcal{F}_A f(n+p) e^{-\frac{i}{2b}(at^2 - 2nt)}, \quad t \in \mathbb{R}.$$

We refer to [7] for the proof.

3 Closed ideals in $(L^1(\mathbb{R}), \star_A)$

We have seen that

$$C_{a/b}(f \star_A g) = \frac{1}{\sqrt{|b|}} (C_{a/b} f \star C_{a/b} g).$$

Thus it is easy to see that $\|g \star_A f\|_1 \leq (2\pi|b|)^{-1/2} \|g\|_1 \|f\|_1$, from which it follows that $(L^1(\mathbb{R}), \star_A)$ is a commutative Banach algebra. In this section, we aim to study the closed ideals in $(L^1(\mathbb{R}), \star_A)$. Towards this end, first we show that $(L^1(\mathbb{R}), \star_A)$ possesses a bounded approximate identity as in the classical case $(L^1(\mathbb{R}), \star)$.

Theorem 3.1 *The Banach algebra $(L^1(\mathbb{R}), \star_A)$ possesses a bounded approximate identity.*

Proof Let $\{g_\alpha\}$ be a bounded approximate identity in $(L^1(\mathbb{R}), \star)$ and define $u_\alpha = \sqrt{|b|} \overline{\rho_A} g_\alpha$. Then using (2.15), we get

$$\begin{aligned} \|f \star_A u_\alpha - f\|_1 &= \|\rho_A(f \star_A u_\alpha) - \rho_A f\|_1 = \left\| \frac{1}{\sqrt{|b|}} ((\rho_A f) \star (\rho_A u_\alpha)) - \rho_A f \right\|_1 \\ &= \|(\rho_A f) \star g_\alpha - \rho_A f\|_1 \rightarrow 0 \end{aligned}$$

as $\alpha \rightarrow \infty$. Hence $\{u_\alpha\}$ is a bounded approximate identity for $(L^1(\mathbb{R}), \star_A)$. \square

Now, we aim to study A -translation invariant closed ideals in $(L^1(\mathbb{R}), \star_A)$. First we prove the following theorem in this context.

Theorem 3.2 *Let J be a closed subspace of $L^1(\mathbb{R})$. Then J is an ideal in $(L^1(\mathbb{R}), \star_A)$ if and only if it is invariant under A -translations.*

Proof Let J be an ideal in $(L^1(\mathbb{R}), \star_A)$. Let $\{u_\alpha\}$ be an approximate identity in $(L^1(\mathbb{R}), \star_A)$. Then for $f \in J$ and $x \in \mathbb{R}$, we have

$$T_x^A f = \lim_{\alpha \rightarrow \infty} T_x^A (u_\alpha \star_A f) = \lim_{\alpha \rightarrow \infty} T_x^A u_\alpha \star_A f,$$

using Theorem 2.6. Since $T_x^A u_\alpha \star_A f \in J$, for all α and J is closed, $T_x^A f \in J$.

Conversely, assume that J is invariant under A -translations. Let $f \in J$, $g \in L^1(\mathbb{R})$. Then viewing $f \star_A g$ as a Bochner integral as in (2.13), we can conclude that $f \star_A g \in J$. \square

Proposition 3.3 *The collection $\{g \in L^1(\mathbb{R}) : \mathcal{F}_A(g) \text{ has compact support}\}$ is dense in $L^1(\mathbb{R})$.*

Proof We know that $\{g \in L^1(\mathbb{R}) : \widehat{g} \text{ has compact support}\}$ is dense in $L^1(\mathbb{R})$. Thus, for $f \in L^1(\mathbb{R})$, there exists $g \in L^1(\mathbb{R})$ such that \widehat{g} has compact support and $\|C_{a/b} f - g\| < \epsilon$. This implies that

$$\|f - C_{-a/b} g\| = \|C_{a/b} f - C_{a/b} C_{-a/b} g\| < \epsilon.$$

Further, $\mathcal{F}_A(C_{-a/b} g)$ has compact support as \widehat{g} has compact support and

$$\mathcal{F}_A(C_{-a/b} g)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} \widehat{g}\left(\frac{\omega - p}{b}\right),$$

which completes the proof. \square

Lemma 3.4 (Lemma 4.59 in [14]) Let $f \in L^1(\mathbb{R})$ and $\omega_0 \in \mathbb{R}$. Then for every $\epsilon > 0$, there exists $h \in L^1(\mathbb{R})$ with $\|h\|_1 < \epsilon$ such that

$$(f + h)^\wedge(\omega) = \widehat{f}(\omega_0),$$

for every ω in some neighbourhood of ω_0 .

Now we are in a position to state and prove an analogue Wiener's theorem in connection with the SAFT.

Theorem 3.5 Let J be a closed A -translation invariant subspace of $L^1(\mathbb{R})$ such that $Z(J) = \emptyset$, where $Z(J) := \{\omega \in \mathbb{R} : \mathcal{F}_A f(\omega) = 0, \text{ for all } f \in J\}$. Then $J = L^1(\mathbb{R})$.

Proof In view of Proposition 3.3, it is enough to show that $f \in J$ for all $f \in L^1(\mathbb{R})$ such that $\mathcal{F}_A f$ has compact support. Let $f \in L^1(\mathbb{R})$ be such that $\mathcal{F}_A f$ has compact support. Let $K = \text{supp}(\mathcal{F}_A f)$.

Step 1: In this step we show that for each $\omega_0 \in \mathbb{R}$, there exists $F \in J$ such that $\mathcal{F}_A f(\omega) = \mathcal{F}_A F(\omega)$ in a neighbourhood of ω_0 . Since $Z(J) = \emptyset$, we can choose $g \in J$ such that $\overline{\eta_A}(\omega_0)\mathcal{F}_A g(\omega_0) = 1$. Then using Lemma 3.4, there exists $h \in L^1(\mathbb{R})$ with $\|h\|_1 < \frac{(2\pi|b|)^{1/2}}{2}$ and

$$(C_{a/b}g)^\wedge\left(\frac{\omega-p}{b}\right) + (C_{a/b}h)^\wedge\left(\frac{\omega-p}{b}\right) = (C_{a/b}g)^\wedge\left(\frac{\omega_0-p}{b}\right),$$

in a neighbourhood of ω_0 . This implies that

$$\mathcal{F}_A g(\omega) + \mathcal{F}_A h(\omega) = \frac{\eta_A(\omega)}{\eta_A(\omega_0)} \mathcal{F}_A g(\omega_0) = \eta_A(\omega), \quad (3.1)$$

in a neighbourhood of ω_0 . Let $h_n = h \star_A h \star_A \cdots \star_A h$ (n -times). Then using the fact that $(L^1(\mathbb{R}), \star_A)$ is a Banach algebra, we can show that the series $f + \sum_{n \in \mathbb{N}} f \star_A h_n$ converges in $L^1(\mathbb{R})$. Let $k = f + \sum_{n \in \mathbb{N}} f \star_A h_n$. Then using convolution theorem, we obtain

$$\begin{aligned} \mathcal{F}_A k(\omega) &= \mathcal{F}_A f(\omega) + \sum_{n \in \mathbb{N}} \overline{\eta_A}(\omega)^n \mathcal{F}_A f(\omega) (\mathcal{F}_A h(\omega))^n = \mathcal{F}_A f(\omega) \frac{1}{1 - \overline{\eta_A}(\omega) \mathcal{F}_A h(\omega)} \\ &= \frac{\mathcal{F}_A f(\omega)}{\overline{\eta_A}(\omega) \mathcal{F}_A g(\omega)}, \end{aligned}$$

in a neighbourhood of ω_0 , using (3.1). The second equality in the above equation follows from the fact that $|\mathcal{F}_A h(\omega)| \leq \frac{1}{(2\pi|b|)^{1/2}} \|h\|_1 < \frac{1}{2}$. Thus

$$\mathcal{F}_A f(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A g(\omega) \mathcal{F}_A k(\omega) = \mathcal{F}_A (g \star_A k)(\omega),$$

in a neighbourhood of ω_0 . As $g \star_A k \in J$, by Theorem 3.2, our claim is established.

Step 2: In this step we show that $f \in J$. Appealing to Step 1, for each $\omega \in \mathbb{R}$, choose $g_\omega \in J$ such that $\mathcal{F}_A f = \mathcal{F}_A g_\omega$ in a neighbourhood U_ω of ω . Using compactness of K , we get U_1, U_2, \dots, U_n and $g_1, g_2, \dots, g_n \in J$ such that $K \subset \cup_{i=1}^n U_i$ and $\mathcal{F}_A f = \mathcal{F}_A g_i$ on U_i . Now choose open W_ω such that

$$\{\omega\} \subset W_\omega \subset \overline{W_\omega} \subset U_i.$$

Again using the compactness of K , there exist $W_{\omega_1}, W_{\omega_2}, \dots, W_{\omega_m}$ such that $K \subset \cup_{i=1}^m \overline{W_{\omega_i}}$ and $\overline{W_{\omega_i}} \subset U_{\omega_i}$, where $\omega_i \in U_{\omega_i} \in \{U_1, U_2, \dots, U_n\}$. Take $h_1, h_2, \dots, h_m \in L^1(\mathbb{R})$ such that

$$\overline{\eta_A}(\omega) \mathcal{F}_A h_j(\omega) = 1 \quad \text{on } \overline{W_{\omega_j}} \quad \text{and } \text{supp}(\mathcal{F}_A h_j) \subset U_{\omega_j}.$$

Then $\prod_{j=1}^m (1 - \overline{\eta_A}(\omega) \mathcal{F}_A h_j(\omega)) = 0$ on K . This implies that

$$\mathcal{F}_A f(\omega) = \mathcal{F}_A f(\omega) [1 - \prod_{j=1}^m (1 - \overline{\eta_A}(\omega) \mathcal{F}_A h_j(\omega))].$$

This can be rewritten as $f = \sum f \star_A H_i$, where H_i being one of the h_j 's or their convolutions, and $\text{supp}(\mathcal{F}_A H_i) \subset U_{\omega_i}$, for some i . But

$$\mathcal{F}_A(f \star_A H_i)(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A f(\omega) \mathcal{F}_A H_i(\omega) = \overline{\eta_A}(\omega) \mathcal{F}_A g_i(\omega) \mathcal{F}_A H_i(\omega) = \mathcal{F}_A(g_i \star_A H_i)(\omega).$$

As $g_i \star_A H_i \in J$, $f \in J$. This completes the proof. \square

Corollary 3.6 *Let $f \in L^1(\mathbb{R})$. Then the closed linear span of A -translates of f is $L^1(\mathbb{R})$ if and only if $\mathcal{F}_A f$ never vanishes.*

Proof Let J be the closed linear span of A -translates of f . Let $\mathcal{F}_A f(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Since J is a closed A -translation invariant subspace, appealing to Wiener's theorem, we get $J = L^1(\mathbb{R})$.

Conversely, assume that $J = L^1(\mathbb{R})$. Suppose $\mathcal{F}_A f(\omega_0) = 0$ for some $\omega_0 \in \mathbb{R}$. Then $\mathcal{F}_A h(\omega_0) = 0$ for all $h \in \text{span}\{T_x^A f : x \in \mathbb{R}\}$. Since $\text{span}\{T_x^A f : x \in \mathbb{R}\}$ is dense in $L^1(\mathbb{R})$ and $\|\mathcal{F}_A h\|_\infty \leq (2\pi|b|)^{-1/2} \|f\|_1$, we can conclude that $\mathcal{F}_A h(\omega_0) = 0$, for all $h \in L^1(\mathbb{R})$, which is an impossibility. Hence the result follows. \square

Now, we shall state the analogue of Wiener's theorem for $L^2(\mathbb{R})$ functions.

Theorem 3.7 *Let $f \in L^2(\mathbb{R})$. Then the closed linear span of A -translates of f is $L^2(\mathbb{R})$ if and only if $\mathcal{F}_A f(\omega) \neq 0$ a.e.*

Proof Let $M = \overline{\text{span}\{T_x^A f : x \in \mathbb{R}\}}$. Then $g \in M^\perp$ if and only if $\langle T_x^A f, g \rangle = 0$, for all $x \in \mathbb{R}$. For $x \in \mathbb{R}$, consider

$$\begin{aligned} \langle T_x^A f, g \rangle &= \langle \mathcal{F}_A(T_x^A f), \mathcal{F}_A g \rangle = \int_{\mathbb{R}} \rho_A(x) e^{-i\frac{x}{b}\omega} \mathcal{F}_A f(\omega) \overline{\mathcal{F}_A g(\omega)} d\omega \\ &= \sqrt{2\pi} \rho_A(x) \left((\mathcal{F}_A f) \overline{(\mathcal{F}_A g)} \right) \wedge \left(\frac{x}{b} \right). \end{aligned}$$

Thus $\langle T_x^A f, g \rangle = 0$ for all $x \in \mathbb{R}$ is equivalent to

$$(\mathcal{F}_A f(\omega) \overline{\mathcal{F}_A g(\omega)}) \wedge (x) = 0, \text{ for all } x \in \mathbb{R},$$

which is same as $\mathcal{F}_A f(\omega) \overline{\mathcal{F}_A g(\omega)} = 0$, a.e. $\omega \in \mathbb{R}$. This shows that $M^\perp = \{0\}$ if and only if $\mathcal{F}_A f(\omega) \neq 0$ a.e. $\omega \in \mathbb{R}$. \square

Example 3.8 Let $\phi(x) = C_{-a/b} g(x)$, where $g(x) = e^{-\frac{x^2}{2}}$. Then using $\mathcal{F}_A(\phi)(\omega) = \frac{\eta_A(\omega)}{\sqrt{2\pi|b|}} (C_{a/b} \phi) \wedge (\frac{\omega-p}{b})$, it follows that

$$\mathcal{F}_A(\phi)(\omega) = \frac{1}{\sqrt{|b|}} g\left(\frac{\omega-p}{b}\right) \neq 0, \text{ for all } \omega \in \mathbb{R}.$$

Thus $\text{span}\{T_x^A \phi : x \in \mathbb{R}\}$ is dense in $L^1(\mathbb{R})$ as well as in $L^2(\mathbb{R})$.

We conclude this section by establishing an analogue of Wiener-Tauberian theorem. Recall that M_y denotes the modulation operator which is defined in (2.6).

Theorem 3.9 (Wiener-Tauberian) Let $\phi \in L^\infty(\mathbb{R})$, $h \in L^1(\mathbb{R})$ be such that $\mathcal{F}_A h(\omega) \neq 0$, for all $\omega \in \mathbb{R}$ and

$$\lim_{x \rightarrow \pm\infty} C_{a/b}(h \star_A \phi)(x) = m \mathcal{F}_A(M_{-\frac{p}{b}} h)(0).$$

Then

$$\lim_{x \rightarrow \pm\infty} C_{a/b}(f \star_A \phi)(x) = m \mathcal{F}_A(M_{-\frac{p}{b}} f)(0),$$

for all $f \in L^1(\mathbb{R})$.

Proof Since, $0 \neq \mathcal{F}_A h(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (C_{a/b} h)^\wedge(\frac{\omega-p}{b})$, for all $\omega \in \mathbb{R}$. $(C_{a/b} h)^\wedge(\omega) \neq 0$, for all $\omega \in \mathbb{R}$. Further,

$$C_{a/b}(h \star_A \phi)(x) = \frac{1}{\sqrt{|b|}} (C_{a/b} h \star C_{a/b} \phi)(x).$$

Furthermore,

$$\begin{aligned} \mathcal{F}_A f(0) &= \frac{1}{\sqrt{|b|}} (C_{a/b} f)^\wedge(-\frac{p}{b}) = \frac{1}{\sqrt{|b|}} T_{\frac{p}{b}}(C_{a/b} f)^\wedge(0) = \frac{1}{\sqrt{|b|}} (M_{\frac{p}{b}} C_{a/b} f)^\wedge(0) \\ &= \frac{1}{\sqrt{|b|}} (C_{a/b} M_{\frac{p}{b}} f)^\wedge(0), \end{aligned}$$

using $C_s M_y = M_y C_s$. Thus

$$\mathcal{F}_A(M_{-\frac{p}{b}} f)(0) = \frac{1}{\sqrt{|b|}} (C_{a/b} f)(0). \quad (3.2)$$

Hence

$$\lim_{x \rightarrow \pm\infty} (C_{a/b} h \star C_{a/b} \phi)(x) = m (C_{a/b} h)^\wedge(0),$$

using given hypothesis. Now using the classical Wiener-Tauberian theorem, we get

$$\lim_{x \rightarrow \pm\infty} (f \star C_{a/b} \phi)(x) = m \hat{f}(0), \quad \text{for all } f \in L^1(\mathbb{R}).$$

This implies that

$$\lim_{x \rightarrow \pm\infty} (C_{a/b} f \star C_{a/b} \phi)(x) = m (C_{a/b} f)^\wedge(0), \quad \text{for all } f \in L^1(\mathbb{R}).$$

This in turn implies that

$$\lim_{x \rightarrow \pm\infty} C_{a/b}(f \star_A \phi)(x) = m \mathcal{F}_A(M_{-\frac{p}{b}} f)(0), \quad \text{for all } f \in L^1(\mathbb{R}),$$

using (3.2). □

As a special case, we state the following analogue of the Wiener-Tauberian theorem associated with the fractional Fourier transform.

Corollary 3.10 Let $\phi \in L^\infty(\mathbb{R})$, $h \in L^1(\mathbb{R})$ be such that $\mathcal{F}_\theta h(\omega) \neq 0$, for all $\omega \in \mathbb{R}$ and

$$\lim_{x \rightarrow \pm\infty} C_\theta(h \star_\theta \phi)(x) = m \mathcal{F}_\theta h(0).$$

Then

$$\lim_{x \rightarrow \pm\infty} C_\theta(f \star_\theta \phi)(x) = m \mathcal{F}_\theta f(0),$$

for all $f \in L^1(\mathbb{R})$.

4 A-shift invariant spaces

In this section, we aim to study shift invariant spaces associated with the SAFT, called A-shift invariant spaces, in detail. Recall that the A-shift invariant space is defined by $V_A(\phi) = \overline{\text{span}}\{T_k^A \phi : k \in \mathbb{Z}\}$ for $\phi \in L^2(\mathbb{R})$. First, we obtain the following result whose proof is similar to that of Theorem 2 in [2] in the classical case.

Theorem 4.1 *Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$ be such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$. Then*

- (i) $V_A(\phi) \subseteq W(C_0, \ell^2)(\mathbb{R})$.
- (ii) *If $\mathcal{X} = \{x_k : k \in \mathbb{Z}\}$ is separated, then there is $M > 0$ such that*

$$\left(\sum_{k \in \mathbb{Z}} |f(x_k)|^2 \right)^{1/2} \leq M \|f\|, \quad \forall f \in V_A(\phi).$$

Corollary 4.2 *Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$ be such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$. Then $V_A(\phi)$ is a reproducing kernel Hilbert space with the reproducing kernel*

$$K(x, y) = \sum_{k \in \mathbb{Z}} e^{i \frac{a}{b} k(x-y)} \overline{\phi(x-k)} S^{-1} \phi(y-k),$$

where S is the frame operator for $\{T_k^A \phi : k \in \mathbb{Z}\}$.

Proof Let $x \in \mathbb{R}$ be fixed. Then, taking $\mathcal{X} = \{x + k : k \in \mathbb{Z}\}$ in the previous theorem we get $M > 0$ such that $|f(x)| \leq M \|f\|$, for every $f \in V_A(\phi)$, which shows that $V_A(\phi)$ is a RKHS. The reproducing kernel for $V_A(\phi)$ is $K(x, y) = \sum_{k \in \mathbb{Z}} \overline{T_k^A \phi(x)} T_k^A S^{-1} \phi(y)$. Now, using the definition of T_k^A our assertion follows. \square

Theorem 4.3 *If $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a frame sequence, for $\phi \in L^2(\mathbb{R})$, then the members of its canonical dual frame also are A-translates of a single function.*

Proof Let S be the frame operator for $\{T_k^A \phi : k \in \mathbb{Z}\}$. First we prove that $ST_k^A = T_k^A S$, $\forall k \in \mathbb{Z}$. Let $f \in V_A(\phi)$. Then for $k \in \mathbb{Z}$, we have

$$\begin{aligned} ST_k^A f &= \sum_{k' \in \mathbb{Z}} \langle T_k^A f, T_{k'}^A \phi \rangle T_{k'}^A \phi \\ &= \sum_{k' \in \mathbb{Z}} \langle f, (T_k^A)^* T_{k'}^A \phi \rangle T_{k'}^A \phi \\ &= \sum_{k' \in \mathbb{Z}} \langle f, e^{-i \frac{a}{b} k^2} T_{-k}^A T_{k'}^A \phi \rangle T_{k'}^A \phi \\ &= \sum_{k' \in \mathbb{Z}} \langle f, T_{k'-k}^A \phi \rangle e^{-i \frac{a}{b} k(k'-k)} T_{k'}^A \phi \\ &= \sum_{\ell \in \mathbb{Z}} \langle f, T_\ell^A \phi \rangle T_k^A T_\ell^A \phi \\ &= T_k^A S f. \end{aligned}$$

Since S is invertible, $T_k^A f = S^{-1} T_k^A S f$ and we have for every $h \in V_A(\phi)$,

$$T_k^A S^{-1} h = S^{-1} T_k^A S S^{-1} h = S^{-1} T_k^A h.$$

Thus, if $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a frame for $V_A(\phi)$, then the canonical dual frame $\{S^{-1}T_k^A \phi : k \in \mathbb{Z}\}$ is given by $\{T_k^A S^{-1} \phi : k \in \mathbb{Z}\}$. Taking $\psi = S^{-1} \phi$, we conclude that the canonical dual frame is also of the form $\{T_k^A \psi : k \in \mathbb{Z}\}$. \square

Remark 4.4 If we assume that the system $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$, then $\{T_k^A S^{-1} \phi : k \in \mathbb{Z}\}$ is the dual Riesz basis of $\{T_k^A \phi : k \in \mathbb{Z}\}$.

Now we obtain a characterization for the system of A -translates $\{T_k^A \phi : k \in \mathbb{Z}\}$ to be a frame sequence in terms of the weight function $w_\phi(\omega) := \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2$.

Theorem 4.5 Let $\phi \in L^2(\mathbb{R})$. Then the system of A -translates $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a frame sequence with bounds $m, M > 0$ if and only if

$$\frac{m}{2\pi|b|} \chi_{E_\phi} \leq \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2 \leq \frac{M}{2\pi|b|} \chi_{E_\phi}, \quad \omega \in I, \quad (4.1)$$

where $E_\phi = \{\omega \in \mathbb{R} : w_\phi(\omega) \neq 0\}$ and $I = [-|b|\pi, |b|\pi]$.

Proof Let $\{T_k^A \phi : k \in \mathbb{Z}\}$ be a frame sequence with bounds $m, M > 0$. Then

$$m\|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, T_k^A \phi \rangle|^2 \leq M\|f\|^2, \quad \text{for all } f \in V_A(\phi). \quad (4.2)$$

Let F be a finite subset of \mathbb{Z} . Let $f = \sum_{k \in F} c_k T_k^A \phi \in V_A(\phi)$. Then $\mathcal{F}_A f(\omega) = r(\omega) \mathcal{F}_A \phi(\omega)$, where $r(\omega) = \sum_{k \in F} c_k \rho_A(k) e^{-\frac{i}{b} k \omega}$. Thus

$$\|f\|^2 = \langle f, f \rangle = \langle \mathcal{F}_A f, \mathcal{F}_A f \rangle = \int_{\mathbb{R}} |r(\omega)|^2 |\mathcal{F}_A \phi|^2 d\omega = \int_I |r(\omega)|^2 w_\phi(\omega) d\omega. \quad (4.3)$$

Similarly,

$$\begin{aligned} \langle f, T_k^A \phi \rangle &= \int_{\mathbb{R}} \mathcal{F}_A f(\omega) \overline{\mathcal{F}_A (T_k^A \phi)(\omega)} d\omega = \int_{\mathbb{R}} r(\omega) |\mathcal{F}_A \phi(\omega)|^2 \overline{\rho_A(k)} e^{\frac{i}{b} k \omega} d\omega \\ &= \int_I r(\omega) \overline{\rho_A(k)} w_\phi(\omega) e^{\frac{i}{b} k \omega} d\omega = \sqrt{2\pi|b|} (r(\omega) w_\phi(\omega))^\wedge(k), \end{aligned}$$

where $\hat{f}(k) = \int_I f(x) \overline{\rho_A(k)} e^{\frac{i}{b} k x} dx$. Hence

$$\sum_{k \in \mathbb{Z}} |\langle f, T_k^A \phi \rangle|^2 = 2\pi|b| \sum_{k \in \mathbb{Z}} |(r(\omega) w_\phi(\omega))^\wedge(k)|^2 = 2\pi|b| \int_I |r(\omega) w_\phi(\omega)|^2 d\omega. \quad (4.4)$$

Now, using (4.2), we get

$$m \int_I |r(\omega)|^2 w_\phi(\omega) d\omega \leq 2\pi|b| \int_I |r(\omega)|^2 (w_\phi(\omega))^2 d\omega \leq M \int_I |r(\omega)|^2 w_\phi(\omega) d\omega,$$

for every $|b|$ -periodic trigonometric polynomial r . This implies that

$$\frac{m}{2\pi|b|} w_\phi(\omega) \leq w_\phi(\omega)^2 \leq \frac{M}{2\pi|b|} w_\phi(\omega),$$

a. e. $\omega \in I$, from which (4.1) follows.

Conversely, assume that (4.1) holds. Then using (4.3), (4.4), we can show that (4.2) holds for all $f \in \text{span}\{T_k^A \phi : k \in \mathbb{Z}\}$. Since $\text{span}\{T_k^A \phi : k \in \mathbb{Z}\}$ is dense in $V_A(\phi)$, the proof follows. \square

In a similar way, we can obtain the characterizations for the system $\{T_k^A \phi : k \in \mathbb{Z}\}$ to be a Riesz sequence or an orthonormal system. We state the results without proof. The interested readers can see the proof form [18] and from [7].

Theorem 4.6 *Let $\phi \in L^2(\mathbb{R})$. Then the collection $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$ if and only if there are $m, M > 0$ such that*

$$m \leq \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2 \leq M, \quad (4.5)$$

for almost all $\omega \in I$.

Remark 4.7 Appealing to (2.4), it follows that $S(\mathbb{R})$, the Schwartz space of rapidly decreasing functions on \mathbb{R} , is invariant under \mathcal{F}_A . See also [9]. As a consequence, it is easy to see that the inequalities on the right-hand side of (4.2) and (4.5) hold for any $\phi \in S(\mathbb{R})$. In other words, for any $\phi \in S(\mathbb{R})$, the system $\{T_k^A \phi : k \in \mathbb{Z}\}$ turns out to be a Bessel sequence.

Theorem 4.8 *Let $\phi \in L^2(\mathbb{R})$. Then the collection $\{T_k^A \phi : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$ if and only if*

$$\sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2 = \frac{1}{2\pi|b|}, \quad (4.6)$$

for almost all $\omega \in I$.

5 Sampling in A-shift invariant spaces

In order to get an equivalent condition for the stable set of sampling in terms of the Zak transform, we first introduce A-Zak transform.

Definition 5.1 The A-Zak transform $\mathcal{Z}_A f$ of a function $f \in L^2(\mathbb{R})$ is a function on \mathbb{R}^2 , defined as

$$\mathcal{Z}_A f(t, \omega) = \frac{\overline{\eta}_A(\omega)}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} T_k^A f(t) e^{-\frac{i}{2b}(ak^2 - 2k\omega + 2pk)}, \quad t, \omega \in \mathbb{R}.$$

One can simplify the right hand side and get

$$\mathcal{Z}_A f(t, \omega) = \frac{\overline{\eta}_A(\omega)}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} f(t - k) e^{\frac{i}{2b}(ak^2 - 2akt + 2k\omega - 2pk)}, \quad \text{for } t, \omega \in \mathbb{R}.$$

Remark 5.2 In particular, if we take $A = \{0, 1, -1, 0, 0, 0\}$, then A-Zak transform reduces to the classical Zak transform.

Theorem 5.3 *The A-Zak transform is an isometry between the spaces $L^2(\mathbb{R})$ and $L^2([0, 1] \times I)$.*

See [7] for the proof.

Define an operator $T : L^2(I) \rightarrow V_A(\phi)$ by

$$TF(x) = \sum_{k \in \mathbb{Z}} \langle F, E_k \rangle T_{-k}^A \phi(x), \quad F \in L^2(I),$$

where $E_k(t) = \frac{\eta_A(t)}{\sqrt{2\pi|b|}} e^{\frac{i}{2b}(ak^2 - 2pk + 2kt)}$. Clearly $\{E_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I)$.

Suppose $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz sequence. Then there are constants $m, M > 0$ such that

$$m \left(\sum_{k \in \mathbb{Z}} |\langle F, E_k \rangle|^2 \right)^{1/2} \leq \left\| \sum_{k \in \mathbb{Z}} \langle F, E_k \rangle T_{-k}^A \phi \right\|_{L^2(\mathbb{R})} \leq M \left(\sum_{k \in \mathbb{Z}} |\langle F, E_k \rangle|^2 \right)^{1/2},$$

for all $F \in L^2(I)$. Since $\{E_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I)$ and $F \in L^2(I)$, the above inequality reduces to

$$m \|F\|_{L^2(I)} \leq \|TF\|_{L^2(\mathbb{R})} \leq M \|F\|_{L^2(I)}, \quad \forall F \in L^2(I).$$

This shows that T is bounded above and bounded below. By Riesz-Fischer theorem T is onto. Hence T is invertible. Moreover, we have

$$TF(x) = \sum_{k \in \mathbb{Z}} \langle F, E_k \rangle T_{-k}^A \phi(x) = \left\langle F, \frac{\eta_A(\cdot)}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} e^{\frac{i}{2b}(ak^2 - 2pk + 2k \cdot)} \overline{T_{-k}^A \phi(x)} \right\rangle = \langle F, \overline{\mathcal{L}_A \phi(x, \cdot)} \rangle. \quad (5.1)$$

Now, we are in a position to prove equivalent conditions for a set to be a stable set of sampling for a shift invariant space $V_A(\phi)$.

Theorem 5.4 Assume that $V_A(\phi)$ is a reproducing kernel Hilbert space, for $\phi \in L^2(\mathbb{R})$, such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$. Then the following statements are equivalent.

- (i) The set $\mathcal{X} = \{x_j : j \in \mathbb{Z}\}$ is a stable set of sampling for $V_A(\phi)$.
- (ii) There are constants $m, M > 0$ such that

$$m \|d\|_{\ell^2(\mathbb{Z})}^2 \leq \|Ud\|_{\ell^2(\mathbb{Z})}^2 \leq M \|d\|_{\ell^2(\mathbb{Z})}^2, \quad \forall d \in \ell^2(\mathbb{Z}),$$

where the operator $U = (U_{j,k})$ is defined by

$$U_{j,k} = T_k^A \phi(x_j) = e^{-i\frac{a}{b}k(x_j - k)} \phi(x_j - k).$$

- (iii) The set of all reproducing kernels $\{K_{x_j} : j \in \mathbb{Z}\}$ for $V_A(\phi)$ is a frame for $V_A(\phi)$.
- (iv) The set $\{\mathcal{L}_A \phi(x_j, \cdot) : j \in \mathbb{Z}\}$ is a frame for $L^2(I)$.

Proof The proof is similar to the classical case. However, in order to make the paper self-contained, we give the outline of the proof. (i) \Leftrightarrow (ii): If $f \in V_A(\phi)$ then there is $d = \{d_k\} \in \ell^2(\mathbb{Z})$ such that $f(x) = \sum_{k \in \mathbb{Z}} d_k T_k^A \phi(x)$, and hence $f(x_j) = \sum_{k \in \mathbb{Z}} d_k T_k^A \phi(x_j) = (Ud)_j$. Since $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$, the following statements are equivalent.

- (a) There are constants $m, M > 0$ such that for every $f \in V_A(\phi)$

$$m \|f\|^2 \leq \sum_{j \in \mathbb{Z}} |f(x_j)|^2 \leq M \|f\|^2. \quad (5.2)$$

- (b) There are constants $m', M' > 0$ such that for $d \in \ell^2(\mathbb{Z})$

$$m' \|d\|^2 \leq \|Ud\|^2 \leq M' \|d\|^2. \quad (5.3)$$

In fact, if (5.2) holds, then

$$m \|f\|^2 \leq \|Ud\|^2 \leq M \|f\|^2.$$

Now by using the fact that $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz sequence, there exist $m'', M'' > 0$ such that $m'' \|d\|^2 \leq \|f\|^2 \leq M'' \|d\|^2$. Thus (5.3) follows from (5.2) with $m' = mm''$ and $M' = MM''$. Similarly, one can prove (5.3) using (5.2).

(i) \Leftrightarrow (iii): Recall that $f(x_j) = \langle f, K_{x_j} \rangle$.

(iii) \Leftrightarrow (iv): Using (5.1) we obtain

$$\sum_{j \in \mathbb{Z}} |f(x_j)|^2 = \sum_{j \in \mathbb{Z}} |\langle f, K_{x_j} \rangle|^2 = \sum_{j \in \mathbb{Z}} |\langle F, \overline{\mathcal{Z}_A \phi(x_j, \cdot)} \rangle|^2,$$

here $TF = f$. Since T is invertible, we get the equivalence of (iii) and (iv). \square

Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$. Then we define the function ϕ_A^\dagger on the interval I , by

$$\phi_A^\dagger(\omega) = \sum_{n \in \mathbb{Z}} \phi(n) \rho_A(n) e^{-\frac{i}{b} n \omega}.$$

From the definition of the A -Zak transform we obtain $\mathcal{Z}_A \phi(0, \omega) = \frac{\overline{\eta_A}(\omega)}{\sqrt{2\pi|b|}} \phi_A^\dagger(\omega)$.

Theorem 5.5 *Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$. Then the operator $U : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ defined by $U_{j,k} = T_k^A \phi(j)$ satisfies the inequalities*

$$\|\phi_A^\dagger\|_0^2 \|d\|^2 \leq \|Ud\|^2 \leq \|\phi_A^\dagger\|_\infty^2 \|d\|^2, \quad \forall d \in \ell^2(\mathbb{Z}),$$

where $\|\phi_A^\dagger\|_0 = \inf_{x \in I} |\phi_A^\dagger(x)|$, $\|\phi_A^\dagger\|_\infty = \sup_{x \in I} |\phi_A^\dagger(x)|$.

Proof Let $d = \{d_n\} \in \ell^2(\mathbb{Z})$. Then

$$(Ud)_n = \sum_{m \in \mathbb{Z}} U_{n,m} d_m = \sum_{m \in \mathbb{Z}} e^{-\frac{i}{b} am(n-m)} \phi(n-m) d_m.$$

Since $\{\frac{1}{\sqrt{2\pi|b|}} \rho_A(n) e^{-\frac{i}{b} n \omega} : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I)$, we have

$$\begin{aligned} 2\pi|b| \|Ud\|^2 &= \int_I \left| \sum_{n \in \mathbb{Z}} (Ud)_n \rho_A(n) e^{-\frac{i}{b} n \omega} \right|^2 d\omega \\ &= \int_I \left| \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} e^{-\frac{i}{b} am(n-m)} \phi(n-m) d_m \rho_A(n) e^{-\frac{i}{b} n \omega} \right|^2 d\omega \\ &= \int_I \left| \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \rho_A(n-m) \phi(n-m) d_m \rho_A(m) e^{-\frac{i}{b} m \omega} \right|^2 d\omega \\ &= \int_I \left| \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \rho_A(n) \phi(n) d_m \rho_A(m) e^{-\frac{i}{b} (m+n) \omega} \right|^2 d\omega \\ &= \int_I |\phi_A^\dagger(\omega)|^2 \left| \sum_{m \in \mathbb{Z}} d_m \rho_A(m) e^{-\frac{i}{b} m \omega} \right|^2 d\omega. \end{aligned}$$

This implies

$$\begin{aligned} \frac{\|\phi_A^\dagger\|_0^2}{2\pi|b|} \int_I \left| \sum_{m \in \mathbb{Z}} d_m \rho_A(m) e^{-\frac{i}{b} m \omega} \right|^2 d\omega &\leq \|Ud\|^2 \\ &\leq \frac{\|\phi_A^\dagger\|_\infty^2}{2\pi|b|} \int_I \left| \sum_{m \in \mathbb{Z}} d_m \rho_A(m) e^{-\frac{i}{b} m \omega} \right|^2 d\omega \end{aligned}$$

or equivalently $\|\phi_A^\dagger\|_0^2 \sum_{m \in \mathbb{Z}} |d_m|^2 \leq \|Ud\|^2 \leq \|\phi_A^\dagger\|_\infty^2 \sum_{m \in \mathbb{Z}} |d_m|^2$, from which the result follows. \square

As a consequence we obtain the following

Corollary 5.6 *Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$ be such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$ and $\phi_A^\dagger(x) \neq 0$ for all $x \in I$. Then \mathbb{Z} is a stable set of sampling for $V_A(\phi)$.*

Proof Since $\phi_A^\dagger(x) \neq 0$ for all $x \in I$, it follows from Theorem 5.5 that U is bounded above and below. Then the assertion follows from Theorem 5.4. \square

We end this section by proving Bernstein type inequality for $V_A(\phi)$. Let \mathcal{A} denote the class of continuously differentiable functions ϕ such that

- (i) $|\phi(x)| \leq M_1|x|^{-0.5-\epsilon}$ and $|\phi'(x)| \leq M_2|x|^{-0.5-\epsilon}$, for sufficiently large x , for some $M_1, M_2, \epsilon > 0$.
- (ii) $\text{ess sup}_{\omega \in I} \sum_{k \in \mathbb{Z}} \left(\frac{\omega+2kb\pi-p}{b}\right)^2 |\mathcal{F}_A \phi(\omega+2kb\pi)|^2 < \infty$.

Theorem 5.7 *Let $\phi \in \mathcal{A}$ be such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$. Then $Bf(x)$ satisfies the following Bernstein type inequality.*

$$\|Bf\|^2 \leq M\|f\|^2, \quad \text{for all } f \in V_A(\phi),$$

where $Bf(x) = f'(x) + \frac{iax}{b} f(x)$ and

$$M = \text{ess sup}_{\omega \in I} \frac{\sum_{k \in \mathbb{Z}} \left(\frac{\omega+2kb\pi-p}{b}\right)^2 |\mathcal{F}_A \phi(\omega+2kb\pi)|^2}{\sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega+2kb\pi)|^2}.$$

Proof Let $f(x) = \sum_{k \in \mathbb{Z}} c_k T_k^A \phi(x)$.

Then $f'(x) = \sum_{k \in \mathbb{Z}} c_k T_k^A \phi'(x) - \sum_{k \in \mathbb{Z}} \frac{iak}{b} c_k T_k^A \phi(x)$. Since $\phi \in \mathcal{A}$, the above equalities hold pointwise. Thus

$$\begin{aligned} Bf(x) &= f'(x) + \frac{iax}{b} f(x) \\ &= \sum_{k \in \mathbb{Z}} c_k T_k^A \phi'(x) - \sum_{k \in \mathbb{Z}} \frac{iak}{b} c_k T_k^A \phi(x) + \frac{iax}{b} \sum_{k \in \mathbb{Z}} c_k T_k^A \phi(x) \\ &= \sum_{k \in \mathbb{Z}} c_k \left(\frac{d}{dx} (T_k^A \phi(x)) + \frac{iax}{b} T_k^A \phi(x) \right) \\ &= \sum_{k \in \mathbb{Z}} c_k B T_k^A \phi(x). \end{aligned}$$

Further, $\mathcal{F}_A f(\omega) = r(\omega) \mathcal{F}_A \phi(\omega)$, where $r(\omega) = \sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b} k \omega}$. Now, using $\mathcal{F}_A(Bf)(\omega) = i \frac{\omega - p}{b} \mathcal{F}_A f(\omega)$ (see Proposition 3.5 in [9]), we obtain

$$\begin{aligned}
 \|Bf\|^2 &= \|\mathcal{F}_A(Bf)\|^2 = \|\mathcal{F}_A(\sum_{k \in \mathbb{Z}} c_k B(T_k^A \phi))\|^2 \\
 &= \int_{\mathbb{R}} |\sum_{k \in \mathbb{Z}} c_k \mathcal{F}_A(BT_k^A \phi)(\omega)|^2 d\omega \\
 &= \int_{\mathbb{R}} |\sum_{k \in \mathbb{Z}} c_k \frac{i(\omega - p)}{b} \mathcal{F}_A(T_k^A \phi)(\omega)|^2 d\omega \\
 &= \int_{\mathbb{R}} |\sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b} k \omega} \frac{\omega - p}{b} \mathcal{F}_A \phi(\omega)|^2 d\omega \\
 &= \int_{\mathbb{R}} |r(\omega)|^2 (\frac{\omega - p}{b})^2 |\mathcal{F}_A \phi(\omega)|^2 d\omega \\
 &= \int_I |r(\omega)|^2 \sum_{k \in \mathbb{Z}} (\frac{\omega + 2kb\pi - p}{b})^2 |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2 d\omega \\
 &\leq M \int_I |r(\omega)|^2 \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2 d\omega \\
 &= M \int_{\mathbb{R}} |r(\omega)|^2 |\mathcal{F}_A \phi(\omega)|^2 d\omega = M \|f\|^2,
 \end{aligned}$$

proving our assertion. \square

6 Sampling theorems

In this section, our aim is to obtain reconstruction formulae for the functions belonging to certain $V_A(\phi)$ from integer samples. We prove sampling formulae with L^2 convergence as well as uniform convergence. As a corollary, we obtain the result proved in [7], namely Shannon sampling theorem for the functions which are bandlimited in the SAFT domain.

Theorem 6.1 *Let $\phi \in W(C_0, \ell^1)(\mathbb{R})$ be such that $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$. Then the following statements are equivalent.*

(i) *There is a function $S \in V_A(\phi)$ such that*

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) T_n^A S(t), \quad \text{for all } f \in V_A(\phi), \quad (6.1)$$

where the series on the right hand side converges unconditionally in $L^2(\mathbb{R})$ and pointwise for all $t \in \mathbb{R}$.

(ii) $1/\phi_A^\dagger \in L^2(I)$.

Proof Assume that there is a $S \in V_A(\phi)$ such that (6.1) holds. Then $\phi = \sum_{n \in \mathbb{Z}} \phi(n) T_n^A S$. Taking SAFT on both sides we obtain

$$\mathcal{F}_A \phi = \sum_{n \in \mathbb{Z}} \phi(n) \mathcal{F}_A(T_n^A S) = \phi_A^\dagger \mathcal{F}_A S. \quad (6.2)$$

We now show that $\Phi_A^\dagger(\omega) \neq 0$, for a.e. $\omega \in \mathbb{R}$. If it were not true, then there would exist a set of positive measure $\Omega \subset \mathbb{R}$ such that $\Phi_A^\dagger(\omega) = 0$, $\omega \in \Omega$. Then $\mathcal{F}_A \phi(\omega) = 0$, $\omega \in \Omega$, using (6.2). Since, Φ_A^\dagger is $2\pi|b|$ -periodic,

$$\mathcal{F}_A \phi(\omega + 2kb\pi) = 0, \quad \omega \in \Omega, \quad k \in \mathbb{Z},$$

which contradicts the fact that $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz sequence, in view of Theorem 4.6. Thus we can rewrite (6.2), as $\mathcal{F}_A S = \mathcal{F}_A \phi / \phi_A^\dagger$. Further,

$$\begin{aligned} \int_{\mathbb{R}} |S(t)|^2 dt &= \int_{\mathbb{R}} |\mathcal{F}_A S(\omega)|^2 d\omega = \int_{\mathbb{R}} \left| \frac{\mathcal{F}_A \phi(\omega)}{\phi_A^\dagger(\omega)} \right|^2 d\omega \\ &= \int_I \frac{\sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2}{|\phi_A^\dagger(\omega)|^2} d\omega \\ &\geq \|G_\phi^A\|_0 \int_I \frac{1}{|\phi_A^\dagger(\omega)|^2} d\omega, \end{aligned}$$

where $\|G_\phi^A\|_0 = \inf_{\omega \in I} \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2$. Since $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz sequence, $\|G_\phi^A\|_0 > 0$. Consequently, $1/\phi_A^\dagger \in L^2(I)$.

Conversely, assume that $1/\phi_A^\dagger \in L^2(I)$. Since $\{\frac{1}{\sqrt{2\pi|b|}} \rho_A(n) e^{-\frac{i}{b} n\omega} : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(I)$, there is a sequence $\{c_n\} \in \ell^2(\mathbb{Z})$ such that

$$\frac{1}{\phi_A^\dagger(\omega)} = \sum_{n \in \mathbb{Z}} c_n \rho_A(n) e^{-\frac{i}{b} n\omega}.$$

Let $F(\omega) = \mathcal{F}_A \phi(\omega) / \phi_A^\dagger(\omega)$. Then

$$\begin{aligned} \int_{\mathbb{R}} |F(\omega)|^2 d\omega &= \int_{\mathbb{R}} \left| \frac{\mathcal{F}_A \phi(\omega)}{\phi_A^\dagger(\omega)} \right|^2 d\omega = \int_I \frac{\sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2}{|\phi_A^\dagger|^2} d\omega \\ &\leq \|G_\phi^A\|_\infty \|1/\phi_A^\dagger\|_{L^2(I)}^2, \end{aligned}$$

where $\|G_\phi^A\|_\infty = \sup_{\omega \in I} \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2$. Since $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz sequence, $\|G_\phi^A\|_\infty < \infty$. Hence $F \in L^2(\mathbb{R})$. Then there is exactly one $S \in L^2(\mathbb{R})$ such that $\mathcal{F}_A S = F$. From the definition of ϕ_A^\dagger we get

$$\mathcal{F}_A S = \mathcal{F}_A \phi \sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b} k \cdot} = \sum_{k \in \mathbb{Z}} c_k \mathcal{F}_A(T_k^A \phi),$$

which shows that $S = \sum_{k \in \mathbb{Z}} c_k T_k^A \phi$ and hence $S \in V_A(\phi)$. Now let $f \in V_A(\phi)$ with representation $f = \sum_{k \in \mathbb{Z}} a_k T_k^A \phi$. Since, $\phi \in W(C_0, \ell^1)(\mathbb{R})$, it follows from Corollary 4.2 that $V_A(\phi)$ is a reproducing kernel Hilbert space. Thus $f(t) = \sum_{k \in \mathbb{Z}} a_k T_k^A \phi(t)$, for all

$t \in \mathbb{R}$. Hence, taking SAFT on both the sides, we obtain

$$\begin{aligned}
 \mathcal{F}_A f &= \sum_{k \in \mathbb{Z}} a_k \mathcal{F}_A(T_k^A \phi) \\
 &= \sum_{k \in \mathbb{Z}} a_k \rho_A(k) e^{-\frac{i}{b}k} \mathcal{F}_A \phi \\
 &= \mathcal{F}_A S \phi_A^\dagger \sum_{k \in \mathbb{Z}} a_k \rho_A(k) e^{-\frac{i}{b}k} \\
 &= \mathcal{F}_A S \sum_{\ell \in \mathbb{Z}} \phi(\ell) \rho_A(\ell) e^{-\frac{i}{b}\ell} \sum_{k \in \mathbb{Z}} a_k \rho_A(k) e^{-\frac{i}{b}k} \\
 &= \mathcal{F}_A S \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_k \phi(\ell) \rho_A(k) \rho_A(\ell) e^{-\frac{i}{b}(\ell+k)} \\
 &= \mathcal{F}_A S \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_k \phi(n-k) \rho_A(k) \rho_A(n-k) e^{-\frac{i}{b}n} \\
 &= \mathcal{F}_A S \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_k \phi(n-k) e^{-\frac{i}{b}k(n-k)} \rho_A(n) e^{-\frac{i}{b}n} \\
 &= \mathcal{F}_A S \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k T_k^A \phi(n) \right) \rho_A(n) e^{-\frac{i}{b}n} \\
 &= \sum_{n \in \mathbb{Z}} f(n) \mathcal{F}_A(T_n^A S). \tag{6.3}
 \end{aligned}$$

The last equality finally gives $f = \sum_{n \in \mathbb{Z}} f(n) T_n^A S$, with convergence in the sense of $L^2(\mathbb{R})$. Now, using the fact that $V_A(\phi)$ is a reproducing kernel Hilbert space, we get that the series in (6.1) convergence pointwise as well. Since, $\phi \in W(C_0, \ell^1)(\mathbb{R})$, the series defining Φ_A^\dagger is unconditionally convergent. Then proceeding as in (6.3), one can show that the series on the right hand side of (6.1) converges unconditionally in $L^2(\mathbb{R})$. \square

In Theorem 6.1 we obtained a sampling formula for functions belonging to A -shift invariant spaces with L^2 convergence. Now, our aim is to obtain another version of a sampling theorem where we obtain both L^2 convergence and pointwise convergence of the corresponding reconstruction formula. Towards this end, we prove the following

Lemma 6.2 *Let $\phi \in L^2(\mathbb{R})$. Then the following statements are equivalent.*

- (i) *For any $\{c_k\} \in \ell^2(\mathbb{Z})$, the series of functions $\sum_{k \in \mathbb{Z}} c_k T_k^A \phi(t)$ converges pointwise to a continuous function.*
- (ii) *$\phi \in C(\mathbb{R})$ and $\sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\phi(t-k)|^2 < \infty$.*

Proof Since $|T_k^A \phi(t)| = |T_k \phi(t)| = |\phi(t-k)|$, the proof follows as in Lemma 1 in [32]. \square

For a sequence $\{c_k\}$, we define

$$\mathcal{F}_A(\{c_k\})(\omega) = \frac{1}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} c[k] e^{\frac{i}{2b}(ak^2 + 2pk - 2\omega k)}.$$

For two sequences $\{c_k\}$ and $\{d_k\}$, we define

$$(\{c_k\} *_A \{d_k\})[n] = \frac{1}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} c[k] T_k^A d[n].$$

Using the fact that $\{\frac{1}{\sqrt{2\pi|b|}}e^{\frac{i}{2b}(ak^2+2pk-2\omega k)} : k \in \mathbb{Z}\}$, is an orthonormal basis for $L^2(I)$, we get

$$\|\mathcal{F}_A\{c_k\}\|^2 = \sum_{k \in \mathbb{Z}} |c[k]|^2.$$

Further, one can show that

$$\mathcal{F}_A\{\{c_k\} * \{d_k\}\}(\omega) = \mathcal{F}_A(\{c_k\})(\omega)\mathcal{F}_A(\{d_k\})(\omega).$$

Thus

$$\int_I |\mathcal{F}_A(\{c_k\})(\omega)|^2 |\mathcal{F}_A(\{d_k\})(\omega)|^2 d\omega = \frac{1}{\sqrt{2\pi|b|}} \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} c[k] T_k^A d[n] \right|^2 \quad (6.4)$$

Theorem 6.3 Let $\{T_k^A \phi : k \in \mathbb{Z}\}$ be a frame for $V_A(\phi)$. Then the following are equivalent.

- (i) The series $\sum_{k \in \mathbb{Z}} c_k T_k^A \phi(t)$ converges pointwise to a continuous function for any $\{c_k\} \in \ell^2(\mathbb{Z})$ and there exists a frame $\{T_k^A \psi : k \in \mathbb{Z}\}$ for $V_A(\phi)$ such that

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) T_k^A \psi(t), \quad \text{for all } f \in V_A(\phi), \quad (6.5)$$

where the series on the right hand side converges unconditionally in $L^2(\mathbb{R})$ and uniformly on \mathbb{R} .

- (ii) $\phi \in C(\mathbb{R})$, $\sum_{k \in \mathbb{Z}} |\phi(t - k)|^2$ is bounded on \mathbb{R} and

$$m \chi_{E_\phi}(\omega) \leq |\phi_A^\dagger(\omega)| \leq M \chi_{E_\phi}(\omega), \quad (6.6)$$

for some $m, M > 0$, where $E_\phi := \{\omega \in \mathbb{R} : w_\phi(\omega) \neq 0\}$, $w_\phi(\omega) := \sum_{k \in \mathbb{Z}} |\mathcal{F}_A \phi(\omega + 2kb\pi)|^2$.

Proof In order to show that (i) implies (ii), it is enough to show that (6.6) holds. Taking $f = \phi$ in (6.5), we get

$$\phi(t) = \sum_{k \in \mathbb{Z}} \phi(k) T_k^A \psi(t).$$

Taking SAFT on both sides we obtain $\mathcal{F}_A \phi(\omega) = \phi_A^\dagger(\omega) \mathcal{F}_A \psi(\omega)$. This implies that $w_\phi(\omega) = |\phi_A^\dagger(\omega)|^2 w_\psi(\omega)$, from which it follows that $E_\phi \subset E_\psi$. Since $\{T_k^A \phi : k \in \mathbb{Z}\}$ and $\{T_k^A \psi : k \in \mathbb{Z}\}$ are frame sequences, there exist $m, M > 0$ such that

$$m \leq |\phi_A^\dagger(\omega)| \leq M, \quad \text{a.e. } \omega \in E_\phi,$$

using Theorem 4.5. We now show that $\phi_A^\dagger(\omega) = 0$ a.e. on $I \setminus E_\phi$. To see this, take $c(\omega) = 1 - \chi_{E_\phi}(\omega)$. Then $c(\omega) = \sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b} k \omega}$, for some $\{c_k\} \in \ell^2(\mathbb{Z})$. Since $c(\omega) \mathcal{F}_A \phi(\omega) = 0$, taking inverse SAFT, we obtain $\sum_{k \in \mathbb{Z}} c_k T_k^A \phi(t) = 0$, for all $t \in \mathbb{R}$. In particular,

$$\{\{c_k\} * \{\phi(k)\}\}[n] = \frac{1}{\sqrt{2\pi|b|}} \sum_{k \in \mathbb{Z}} c_k T_k^A \phi(n) = 0.$$

Thus using (6.4), we get

$$0 = \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} c_k T_k^A \phi(n) \right|^2 = \int_I |\mathcal{F}_A\{c_k\}(\omega)|^2 |\mathcal{F}_A\{\phi(n)\}(\omega)|^2 d\omega = \int_{I \setminus E_\phi} |\phi_A^\dagger(\omega)|^2 d\omega,$$

which proves our claim.

Conversely assume that (ii) holds. Let

$$\mathcal{F}_A \psi(\omega) = \begin{cases} \frac{1}{\phi_A^\dagger(\omega)} \mathcal{F}_A \phi(\omega) & \omega \in E_\phi \\ 0, & \omega \notin E_\phi. \end{cases}$$

Then $\{T_k^A \psi : k \in \mathbb{Z}\}$ is a frame sequence by appealing to Theorem 4.5. Since $\frac{1}{\phi_A^\dagger} \in L^2(I)$, it can be easily seen that $\psi \in V_A(\phi)$. With the similar reasoning, we can say that $\phi \in V_A(\psi)$. Thus $V_A(\phi) = V_A(\psi)$. Now define,

$$\mathcal{F}_A \tilde{\psi}(\omega) = \begin{cases} \frac{\phi_A^\dagger(\omega)}{w_\phi(\omega)} \mathcal{F}_A \phi(\omega), & \omega \in E_\phi \\ 0, & \omega \notin E_\phi. \end{cases}$$

We show that $\{T_k^A \tilde{\psi} : k \in \mathbb{Z}\}$ is the canonical dual of $\{T_k^A \psi : k \in \mathbb{Z}\}$. Let S be the frame operator associated with the frame $\{T_k^A \psi : k \in \mathbb{Z}\}$. Since S commutes with T_k^A , for all k , it is enough to show that $S\tilde{\psi} = \psi$. Consider

$$\begin{aligned} \mathcal{F}_A(S\tilde{\psi}) &= \sum_{k \in \mathbb{Z}} \langle \tilde{\psi}, T_k^A \psi \rangle \mathcal{F}_A(T_k^A \psi) \\ &= \sum_{k \in \mathbb{Z}} \rho_A(k) e^{-\frac{i}{b}k \cdot} (\mathcal{F}_A \psi) \langle \mathcal{F}_A \tilde{\psi}, \mathcal{F}_A(T_k^A \psi) \rangle \\ &= (\mathcal{F}_A \psi) \sum_{k \in \mathbb{Z}} \rho_A(k) e^{-\frac{i}{b}k \cdot} \int_{\mathbb{R}} \mathcal{F}_A \tilde{\psi}(\eta) \overline{\mathcal{F}_A(T_k^A \psi)(\eta)} d\eta \\ &= (\mathcal{F}_A \psi) \sum_{k \in \mathbb{Z}} \rho_A(k) e^{-\frac{i}{b}k \cdot} \int_{\mathbb{R}} \mathcal{F}_A \tilde{\psi}(\eta) \overline{\rho_A(k)} e^{\frac{i}{b}k\eta} \overline{\mathcal{F}_A \psi(\eta)} d\eta \\ &= (\mathcal{F}_A \psi) \frac{1}{2\pi|b|} \sum_{k \in \mathbb{Z}} e^{-\frac{i}{b}k \cdot} \int_{E_\phi} e^{\frac{i}{b}k\eta} \frac{|\mathcal{F}_A \phi(\eta)|^2}{w_\phi(\eta)} d\eta \\ &= (\mathcal{F}_A \psi) \frac{1}{2\pi|b|} \sum_{k \in \mathbb{Z}} e^{-\frac{i}{b}k \cdot} \int_I \chi_{E_\phi}(\eta) e^{\frac{i}{b}k\eta} d\eta \\ &= (\mathcal{F}_A \psi) \chi_{E_\phi} = \mathcal{F}_A \psi, \end{aligned}$$

which proves our claim. Let $f \in V_A(\phi)$. Then $\mathcal{F}_A f(\omega) = r(\omega) \mathcal{F}_A \phi(\omega)$, where $r(\omega) = \sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b}k\omega}$, for some $\{c_k\} \in \ell^2(\mathbb{Z})$. For $k \in \mathbb{Z}$, consider

$$\begin{aligned} \langle f, T_k^A \tilde{\psi} \rangle &= \int_{\mathbb{R}} \mathcal{F}_A f(\omega) \overline{\mathcal{F}_A(T_k^A \tilde{\psi})(\omega)} d\omega \\ &= \frac{1}{2\pi|b|} \int_{E_\phi} r(\omega) |\mathcal{F}_A \phi(\omega)|^2 \frac{\phi_A^\dagger(\omega)}{w_\phi(\omega)} \overline{\rho_A(k)} e^{\frac{i}{b}k\omega} d\omega \\ &= \frac{1}{2\pi|b|} \int_I r(\omega) \phi_A^\dagger(\omega) \overline{\rho_A(k)} e^{\frac{i}{b}k\omega} d\omega \\ &= \int_I \mathcal{F}_A \{c_k\}(\omega) \mathcal{F}_A \{\phi(n)\}(\omega) \overline{\rho_A(k)} e^{\frac{i}{b}k\omega} d\omega \\ &= \int_I \mathcal{F}_A \{\{c_k\} * \{\phi(n)\}\}(\omega) \overline{\rho_A(k)} e^{\frac{i}{b}k\omega} d\omega \\ &= \sqrt{2\pi|b|} \{\{c_k\} * \{\phi(n)\}\} = f(k). \end{aligned}$$

Hence

$$f(t) = \sum_{k \in \mathbb{Z}} \langle f, T_k^A \tilde{\psi} \rangle T_k^A \psi(t) = \sum_{k \in \mathbb{Z}} f(k) T_k^A(\psi)(t). \quad (6.7)$$

Now, the unconditional $L^2(\mathbb{R})$ convergence follows from Theorem 5.1.6 in [12]. Further, the pointwise convergence follows from Lemma 6.2 and the given hypothesis. Furthermore, observe that $\{f(k)\} = \{\langle f, T_k^A \tilde{\psi} \rangle\} \in \ell^2(\mathbb{Z})$. Thus, in order to show the uniform convergence in (6.7), it is enough to show that

$$\sum_{k \in \mathbb{Z}} |T_k^A \psi(t)|^2 = \sum_{k \in \mathbb{Z}} |\psi(t - k)|^2 \leq M < \infty, \quad \text{for all } t \in \mathbb{R},$$

for some $M > 0$. Since $\psi \in V_A(\phi)$, $\mathcal{F}_A \psi(\omega) = r(\omega) \mathcal{F}_A \phi(\omega)$, where $r(\omega) = \sum_{k \in \mathbb{Z}} c_k \rho_A(k) e^{-\frac{i}{b} k \omega}$, for some $\{c_k\} \in \ell^2(\mathbb{Z})$. Thus $w_\psi(\omega) = |r(\omega)|^2 w_\phi(\omega)$. Since $\{T_k^A \phi : k \in \mathbb{Z}\}$ and $\{T_k^A \psi : k \in \mathbb{Z}\}$ are frames for $V_A(\phi)$, r is bounded on E_ϕ . This implies that $\tilde{r}(\omega) = r(\omega) \chi_{E_\phi}(\omega)$ is bounded on I . Let $\tilde{r}(\omega) = \sum_{k \in \mathbb{Z}} \tilde{c}_k \rho_A(k) e^{-\frac{i}{b} k \omega}$, for some $\{\tilde{c}_k\} \in \ell^2(\mathbb{Z})$. Since $r(\omega) \mathcal{F}_A \phi(\omega) = \tilde{r}(\omega) \mathcal{F}_A \phi(\omega)$, $\psi(t) = \sum_{k \in \mathbb{Z}} \tilde{c}_k T_k^A \phi(t)$. Thus

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\psi(t - n)|^2 &= \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \tilde{c}_k T_k^A \phi(t - n) \right|^2 \\ &= \frac{1}{2\pi|b|} \int_I |\tilde{r}(\omega)|^2 \left| \sum_{n \in \mathbb{Z}} \phi(t - n) \rho_A(n) e^{-\frac{i}{b} n \omega} \right|^2 d\omega \\ &\leq \|\tilde{r}(\omega)\|_\infty \sum_{n \in \mathbb{Z}} |\phi(t - n)|^2, \end{aligned}$$

proving our assertion. \square

As a consequence of Theorem 6.1, we obtain a SAFT version of the Shannon sampling theorem by taking $\phi = C_{-a/b} \text{sinc}$, where $\text{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$. We also write down the sampling theorem when ϕ is taken to be the second order symmetric B -spline.

Corollary 6.4 *Let $\phi = \text{sinc}$ and $\psi = C_{-a/b} \phi$. Then for every $f \in V_A(\psi)$, we have the following representation*

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) e^{i \frac{a}{2b} (k^2 - t^2)} \text{sinc}(t - k), \quad t \in \mathbb{R}, \quad (6.8)$$

where the above series converges unconditionally in $L^2(\mathbb{R})$ and uniformly on \mathbb{R} .

Proof We have

$$\begin{aligned} \mathcal{F}_A \psi(\omega) &= \frac{\eta_A(\omega)}{\sqrt{2\pi|b|}} \int_{\mathbb{R}} C_{-a/b} \phi(t) e^{\frac{i}{2b} (at^2 + 2pt - 2\omega t)} dt \\ &= \frac{\eta_A(\omega)}{\sqrt{|b|}} \hat{\phi} \left(\frac{\omega - p}{b} \right) \\ &= \frac{\eta_A(\omega)}{\sqrt{2\pi|b|}} \chi_{[-\pi, \pi]} \left(\frac{\omega - p}{b} \right) \\ &= \frac{\eta_A(\omega)}{\sqrt{2\pi|b|}} \chi_{I+p}(\omega), \end{aligned}$$

from which it follows that $\sum_{k \in \mathbb{Z}} |\mathcal{F}_A \psi(\omega + 2kb\pi)|^2 = \frac{1}{2\pi|b|}$ for almost all $\omega \in I$, and this implies that $\{T_k^A \psi : k \in \mathbb{Z}\}$ is an orthonormal basis for the space $V_A(\psi)$. Moreover, we have $\psi_A^\dagger(\omega) = \sum_{k \in \mathbb{Z}} C_{-a/b} \text{sinc}(k) \rho_A(k) e^{-\frac{i}{b}k\omega} = 1$. Consequently, $1/\psi_A^\dagger = 1$. So by Theorem 6.1 we have $\mathcal{F}_A S(\omega) = \mathcal{F}_A \psi(\omega)$, which implies that $S = \psi$. Hence for $f \in V_A(\psi)$ we have

$$\begin{aligned} f(t) &= \sum_{k \in \mathbb{Z}} f(k) T_k^A \psi(t) = \sum_{k \in \mathbb{Z}} f(k) e^{-i \frac{a}{b} k(t-k)} e^{i \frac{a}{2b}(t-k)^2} \text{sinc}(t-k) \\ &= \sum_{k \in \mathbb{Z}} f(k) e^{-i \frac{a}{2b}(t^2-k^2)} \text{sinc}(t-k). \end{aligned}$$

□

Corollary 6.5 *Let $\phi = \text{sinc}$. Let $\psi = C_{-a/b}\phi$. Then $V_A(\psi) = B_{I+p}^A$, where $B_{I+p}^A = \{f \in L^2(\mathbb{R}) : \text{supp } \mathcal{F}_A(f) \subseteq I + p\}$.*

Proof It is clear from Corollary 6.4 that $V_A(\psi) \subseteq B_{I+p}^A$. Now we shall show that B_{I+p}^A is a closed subspace of $L^2(\mathbb{R})$ and the orthogonal complement of $V_A(\psi)$ in B_{I+p}^A is zero, which will prove our assertion. As we have $\mathcal{F}_A(C_{-a/b}f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} \hat{f}(\frac{\omega-p}{b})$, $f \mapsto C_{-a/b}f$ is an isometry from $B_{[-\pi, \pi]}(f) := \{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subset [-\pi, \pi]\}$ onto B_{I+p}^A . Therefore B_{I+p}^A is a closed subspace of $L^2(\mathbb{R})$. In Corollary 6.4 we have seen that $\{T_k^A \psi : k \in \mathbb{Z}\}$ is an orthonormal basis for $V_A(\psi)$. Consider $f \in B_{I+p}^A$ such that $\langle f, T_k^A \psi \rangle = 0$, $\forall k \in \mathbb{Z}$. Now we shall show that $f = 0$. For all $k \in \mathbb{Z}$ we have

$$\begin{aligned} 0 &= \langle f, T_k^A \psi \rangle = \langle \mathcal{F}_A(f), \mathcal{F}_A(T_k^A \psi) \rangle \\ &= \int_{I+p} \mathcal{F}_A(f)(\omega) \overline{\rho_A(k)} e^{\frac{i}{b}k\omega} \overline{\mathcal{F}_A(\psi)(\omega)} d\omega \\ &= \frac{\overline{\rho_A(k)}}{\sqrt{2\pi|b|}} \int_{I+p} \mathcal{F}_A(f)(\omega) \overline{\eta_A(\omega)} e^{\frac{i}{b}k\omega} d\omega. \end{aligned}$$

Using the fact that $\{\frac{1}{\sqrt{2\pi|b|}} e^{-\frac{i}{b}k\omega} : k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(I+p)$, we get $\mathcal{F}_A(f)(\omega) \overline{\eta_A(\omega)} = 0$ for a.e. $\omega \in I+p$, which in turn implies that $\|f\|_2 = \|\mathcal{F}_A(f)\|_2 = 0$, proving our assertion. □

As a consequence of Corollary 6.4, Corollary 6.5 can be restated as a SAFT version of the Shannon sampling theorem.

Corollary 6.6 *Let $f \in L^2(\mathbb{R})$ be such that the $\text{supp}(\mathcal{F}_A f) \subseteq I + p$. Then the following sampling formula holds*

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) e^{i \frac{a}{2b}(k^2-t^2)} \text{sinc}(t-k), \quad t \in \mathbb{R},$$

where the above series converges unconditionally in $L^2(\mathbb{R})$ and uniformly on \mathbb{R} .

Corollary 6.7 *Let $\phi = \chi_{[0,1]} \star \chi_{[0,1]}$. Then for every $f \in V_A(\phi)$, we have the following reconstruction formula*

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) T_n^A \phi(t), \quad t \in \mathbb{R},$$

where the above series converges unconditionally in $L^2(\mathbb{R})$ and uniformly on \mathbb{R} .

Proof Let G be the Gramian associated with the sequence $\{T_k^A \phi : k \in \mathbb{Z}\}$. Then $G_{j,k} = \langle T_k^A \phi, T_j^A \phi \rangle$ is a tridiagonal operator with all the diagonal elements, $d_i = 1$. The elements above the diagonal are given by

$$u_j = \begin{cases} \frac{b^2}{a^2} e^{i \frac{a}{b} j} (e^{-i \frac{a}{b}} (\frac{2ib}{a} - 1) - (1 + \frac{2ib}{a})), & a \neq 0 \\ 1/6 & a = 0, \end{cases}$$

and the elements below the diagonal are given by

$$l_j = \begin{cases} -\frac{b^2}{a^2} e^{-i \frac{a}{b} j} (e^{i \frac{a}{b}} (\frac{2ib}{a} + 1) + (1 - \frac{2ib}{a})), & a \neq 0 \\ 1/6 & a = 0. \end{cases}$$

Notice that for $a = 0$, G is strictly diagonally dominant and hence invertible, for $a \neq 0$ $|u_j|$, $|l_j|$ are dominated by $\frac{4b^3}{a^3} \sqrt{\frac{a^2}{4b^2} + 1}$. Now let $\frac{a}{2b} = r$ with $|r| \geq 1.2$, then G is strictly diagonally dominant and hence invertible. In other words, $\{T_k^A \phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V_A(\phi)$. Further $\phi_A^\dagger = 1$ on I , from which the required assertion follows. \square

7 A local reconstruction method

As in the case of classical shift invariant space we can obtain a local reconstruction method for functions belonging to $V_A(\phi)$ with continuous generators satisfying polynomial decay from their samples. We state the results without the proof as the proofs follow similar lines. We refer to the works [23, 24].

Proposition 7.1 *Let ϕ be a complex valued continuous function on \mathbb{R} satisfying $\phi(x) = o(|x|^{-\rho})$ ($\rho > 1$). Assume that $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$. Let $f \in V_A(\phi)$ and $[a', b']$ be an interval in \mathbb{R} . Then for a given $\epsilon > 0$ there exist a positive integer M and a sequence $c_f = \{c_k\} \in \ell^2(\mathbb{Z})$ such that*

$$|f(x) - g_r(x)| < \|c_f\|_{\ell^2(\mathbb{Z})} \frac{\epsilon}{N^{\frac{\rho}{2}}},$$

for all $N \geq M$, for all $x \in [a', b']$ and $g_r(x) = \sum_{k \in [a' - N + 1, b' + N - 1]} c_k T_k^A \phi(x)$. In other words, $f|_{[a', b']}$ can be reconstructed locally with a finite number of coefficients c_k .

Theorem 7.2 *Fix $\rho \geq 2$. Let ϕ be a complex valued continuous function on \mathbb{R} satisfying $\phi(x) = o(|x|^{-\rho})$. Assume that $\{T_k^A \phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V_A(\phi)$. Let $f \in V_A(\phi)$, $[a', b']$ be an interval in \mathbb{R} and $\epsilon > 0$. Let M be a positive integer obtained in Proposition 7.1. Consider those points x_j in the sample set X such that $x_j \in [a', b']$. Let $(2M + b' - a' - 1) \leq \#X \leq M^\rho$, where $\#X$ denotes the number of points in X . Define $U_{j,k} = T_k^A \phi(x_j)$, $1 \leq j \leq \#X$, $k \in [a' - M + 1, b' + M - 1] \cap \mathbb{Z}$. Then there exist $g_r \in V_A(\phi)$ such that*

$$\|f|_X - g_r|_X\| \leq \epsilon(1 + \|U\| \|U^\dagger\|) + \mathcal{O}(\epsilon^2),$$

where U^\dagger is the pseudoinverse of U .

Remark 7.3 The implementation of the local reconstruction method using Mathematica can be found in the thesis [8]. There are experimental confirmations of the theoretical results obtained in this section. Here we omit the details.

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Declarations

Conflict of interest All authors read and approved the final manuscript. There is no Conflict of interest.

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