COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **13**, Number **3**, May **2014**

doi:10.3934/cpaa.2014.13.1141

pp. **1141–1165**

PULLBACK EXPONENTIAL ATTRACTORS FOR EVOLUTION PROCESSES IN BANACH SPACES: PROPERTIES AND APPLICATIONS

Alexandre N. Carvalho

Instituto de Ciências Matemticas e de Computaçao
Universidade de São Paulo - Campus São Carlos, Caixa Postal 668
13560-970 São Carlos SP, Brasil

Stefanie Sonner

BCAM Basque Center for Applied Mathematics
Mazarredo 14, E-48009 Bilbao, Basque Country, Spain

and Helmholtz Zentrum München, Institut für Biomathematik und Biometrie
Ingolstädter Landstrasse 1, D-85764 Neuherberg, Germany

(Communicated by Tohru Ozawa)

Abstract. This article is a continuation of our previous work [5], where we formulated general existence theorems for pullback exponential attractors for asymptotically compact evolution processes in Banach spaces and discussed its implications in the autonomous case. We now study properties of the attractors and use our theoretical results to prove the existence of pullback exponential attractors in two examples, where previous results do not apply.

1. **Introduction.** Global pullback attractors proved to be a useful tool to study the asymptotic dynamics of infinite dimensional non-autonomous dynamical systems. To be more precise, let here and in the sequel (X, d_X) denote a complete metric space and $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. The rules of time evolution in the non-autonomous setting are dictated by a two-parameter family $U = \{U(t,s)|\ t \geq s\}$, $t, s \in \mathbb{T}$, of continuous operators from X into itself, which is called an ***evolution process*** in X, if it satisfies the properties

$$\begin{aligned} &U(t,t) = Id && t \in \mathbb{T}, \\ &U(t,s) \circ U(s,r) = U(t,r) && t \geq s \geq r,\ t,s,r \in \mathbb{T} \\ &(t,s,x) \mapsto U(t,s)x && \text{is continuous from } \mathcal{T} \times X \to X, \end{aligned}$$

where $\mathcal{T} := \{(t,s) \in \mathbb{T} \times \mathbb{T}|\ t \geq s\}$, Id denotes the identity in X and $\circ$ the composition of operators.

2010 *Mathematics Subject Classification.* Primary: 37L25, 37B55; Secondary: 37L30, 35B40.

Key words and phrases. Exponential attractors, non-autonomous dynamical systems, pullback and forwards attractors, non-autonomous Chafee-Infante equation, non-autonomous damped wave equation.

The first author is supported by CNPq 302022/2008-2, CAPES/DGU 267/2008 and FAPESP 2008/55516-3, Brazil. The second author is supported by the ERC Advanced Grant FPT-246775 NUMERIWAVES..

If $\mathbb{T} = \mathbb{Z}$ we call U a *discrete evolution process* and for $\mathbb{T} = \mathbb{R}$ a *time continuous evolution process*.

Definition 1.1. The family of non-empty subsets $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ of X is called the ***global pullback attractor of the evolution process*** $\{U(t,s)|\ t \geq s\}$ if the sets $\mathcal{A}(t)$ are compact, for all $t \in \mathbb{T}$, and the family $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ is strictly invariant,

$$U(t,s)\mathcal{A}(s) = \mathcal{A}(t) \qquad \forall\, t \geq s.$$

Moreover, it pullback attracts all bounded subsets of X; that is, for every time $t \in \mathbb{T}$ the set $\mathcal{A}(t)$ pullback attracts every bounded set $D \subset X$ at time t,

$$\lim_{s\to\infty} \mathrm{dist}_{\mathrm{H}}\big(U(t,t-s)D, \mathcal{A}(t)\big) = 0,$$

and $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ is minimal within the families of closed subsets that pullback attract all bounded subsets of X.

Here, $\mathrm{dist}_{\mathrm{H}}(\cdot,\cdot)$ is the Hausdorff semidistance in X; that is,

$$\mathrm{dist}_{\mathrm{H}}(A,B) = \sup_{a\in A} \inf_{b\in B} d_X(a,b) \qquad \text{for subsets } A, B \subset X.$$

Different from the definition of global attractors in the autonomous case, the minimality property is an additional property needed to ensure the uniqueness of the global pullback attractor. It can be omitted if the pullback attractor is uniformly bounded in the past, i.e., if the union

$$\bigcup_{t\leq t_0} \mathcal{A}(t)$$

is bounded for all $t_0 \in \mathbb{T}$. The following theorem characterizes the evolution processes possessing a global pullback attractor, for its proof we refer to [7] or [4].

Theorem 1.2. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in a complete metric space X. Then, the following statements are equivalent:*

(*a*) *The evolution process $\{U(t,s)|\ t \geq s\}$ possesses a global pullback attractor.*

(*b*) *There exists a family of compact subsets $\{K(t)|\ t \in \mathbb{T}\}$ of X such that for all $t \in \mathbb{T}$ the set $K(t)$ pullback attracts all bounded subsets of X at time t.*

Furthermore, the pullback global attractor is given by

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\,\subset\, X \\ \text{bounded}}} \omega(D,t)} \qquad t \in \mathbb{T},$$

where $\omega(D,t)$ denotes the pullback ω-limit set of the set $D \subset X$ at time instant $t \in \mathbb{T}$.

The ***pullback ω-limit set*** of the subset $D \subset X$ at time instant $t \in \mathbb{T}$ is defined by

$$\omega(D,t) := \bigcap_{r\geq 0} \overline{\bigcup_{s\geq r} U(t,t-s)D},$$

and $\overline{A}$ denotes the closure of a subset $A \subset X$.

Like global attractors of semigroups in the autonomous context, global pullback attractors are generally not stable under perturbations and the rate of convergence to the attractor is unknown, which motivates to consider *pullback exponential attractors* (see [8], [16] and [5]). Pullback exponential attractors are families of compact subsets of the phase space whose fractal dimension is uniformly bounded and that pullback attract all bounded sets at an exponential rate. They are, due to the exponential rate of attraction, more stable under perturbations and contain the global pullback attractor. In particular, to show the existence of a pullback exponential attractor is one way of proving the existence and finite dimensionality of the global pullback attractor.

Definition 1.3. Let $\{U(t,s) |\ t \geq s\}$ be an evolution process in the metric space (X, d_X). We call the family of non-autonomous sets $\mathcal{M} = \{\mathcal{M}(t) |\ t \in \mathbb{T}\}$ a ***pullback exponential attractor for the evolution process*** $\{U(t,s) |\ t \geq s\}$ if

(i) the subset $\mathcal{M}(t) \subset X$ is non-empty and compact $\forall\, t \in \mathbb{T}$,

(ii) the family $\mathcal{M}$ is positively semi-invariant; that is,
$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t) \qquad \forall\, t \geq s,$$

(iii) the fractal dimension of the sections $\mathcal{M}(t)$, $t \in \mathbb{T}$, is uniformly bounded,
$$\sup_{t \in \mathbb{T}} \left\{ \dim_{\mathrm{f}}^{X}(\mathcal{M}(t)) \right\} < \infty,$$

(iv) and $\mathcal{M}$ exponentially pullback attracts all bounded subsets of X: There exists a positive constant $\omega > 0$ such that for every bounded subset $D \subset X$ and every $t \in \mathbb{T}$
$$\lim_{s \to \infty} e^{\omega s} \mathrm{dist}_{\mathrm{H}}\big(U(t, t-s)D, \mathcal{M}(t)\big) = 0.$$

If an evolution process possesses a pullback exponential attractor $\{\mathcal{M}(t) |\ t \in \mathbb{T}\}$, the existence of the global pullback attractor $\{\mathcal{A}(t) |\ t \in \mathbb{T}\}$ follows immediately from Theorem 1.2. Moreover, the global pullback attractor is contained in the pullback exponential attractor and possesses finite dimensional sections. Indeed, the minimality property in Definition 1.1 implies
$$\mathcal{A}(t) \subset \mathcal{M}(t) \qquad \forall t \in \mathbb{T}.$$

An algorithm for the construction of non-autonomous exponential attractors was first developed in [12] for discrete evolution processes, where the authors considered forwards exponential attractors. The method is based on the compact embedding of the phase space V into an auxiliary normed space W and the *smoothing* or *regularizing property* of the evolution process (see Section 2). Using the pullback approach the result was recently extended in [8] and [16] for time continuous evolution processes. Common assumptions in both articles were that the process satisfies the smoothing property, which implies that it is eventually compact, and the existence of a fixed bounded uniformly pullback absorbing set. This allows the pullback exponential attractor $\mathcal{M}$ to be unbounded in the future but it is always uniformly bounded in the past, i.e., the union
$$\bigcup_{t \leq t_0} \mathcal{M}(t)$$
is bounded for all $t_0 \in \mathbb{T}$. Moreover, the Hölder continuity in time of the evolution process was essential for the construction in [8] and [16]. It is typically satisfied in parabolic problems, but not by evolution processes generated by hyperbolic equations. We proposed an alternative method for time-continuous evolution processes in [5], which does not require the Hölder continuity in time of the evolution process, we extended the algorithm for evolution processes that are asymptotically compact and considered a time-dependent family of bounded pullback absorbing sets instead of a fixed bounded pullback absorbing set. Our construction leads to better bounds

for the fractal dimension of the sections of the attractors and to existence results for pullback exponential attractors that are not necessarily uniformly bounded in the past. To prove the finite fractal dimension of global pullback attractors that are not uniformly bounded in the past has been an open problem. Previous constructions of pullback exponential attractors were therefore limited to evolution processes possessing global pullback attractors that are uniformly bounded in the past (see Section 1 in [16] and Remark 3.2 in [18]).

In [13] the authors proposed a construction for forwards exponential attractors for time continuous evolution processes, which is similar to our method. However, the existence of the uniform attractor for the evolution process is a priori known and the existence of a fixed bounded uniformly forwards absorbing set is assumed. This is equivalent to the assumption of a fixed bounded uniformly pullback absorbing set and implies the uniform boundedness of the forwards exponential attractor (i.e., $\bigcup_{t\in\mathbb{T}}\mathcal{M}(t)$ is bounded). They consider asymptotically compact evolution processes in the weaker space W, a construction for processes that are asymptotically compact in the stronger phase space V as we formulated in [5] has not been considered before (see [8], [16], [13] and for autonomous exponential attractors [11], [12]). We discussed and compared these different settings and results in [5], Section 3.2.

Our present article is the continuation of [5], where we constructed pullback exponential attractors for asymptotically compact evolution processes in Banach spaces assuming that the process possesses a family of time-dependent pullback absorbing sets that possibly grow in the past and studied its implications in the autonomous setting. We now discuss properties of the attractors and apply the theoretical results to prove the existence of pullback exponential attractors in two applications. In both examples, previous results are not applicable and the generalizations we developed in [5] are essentially needed.

In particular, we consider a non-autonomous Chafee-Infante equation in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$,

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \triangle u(x,t)+\lambda u(x,t)-\beta(t)\big(u(x,t)\big)^3 && x\in\Omega,\ t>s,\\
\frac{\partial}{\partial \nu}u(x,t) &= 0 && x\in\partial\Omega,\ t\geq s,\\
u(x,s) &= u_s(x) && x\in\Omega,\ s\in\mathbb{R},
\end{aligned}$$

where $\lambda>0$ and the initial data $u_s\in C(\overline{\Omega})$. The non-autonomous term $\beta:\mathbb{R}\to\mathbb{R}_+$ is strictly positive, continuously differentiable, bounded when time t tends to ∞ and vanishes as t goes to $-\infty$. We show that the generated evolution process satisfies the smoothing property and possesses a semi-invariant family of pullback absorbing sets. The diameter of the absorbing sets grows in the past since the function β vanishes when t tends to $-\infty$. From our results in [5] we deduce the existence of a pullback exponential attractor for the generated evolution process. This implies that the global pullback attractor exists and that its sections are of finite fractal dimension. Furthermore, we prove that the global pullback attractor is unbounded in the past,

$$\lim_{t\to-\infty}\operatorname{diam}(\mathcal{A}(t))=\infty,$$

where diam denotes the diameter in the space $C(\overline{\Omega})$, which provides a positive answer to the question whether the finite fractal dimension can be established for global pullback attractors that are not uniformly bounded in the past (see Section 1 in [16] and Remark 3.2 in [18]).

The second application is the non-autonomous dissipative wave equation

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}u(x,t) + \beta(t)\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) + f(u(x,t)) && x \in \Omega,\ t > s,\\
&u(x,t) = 0 && x \in \partial\Omega,\ t \geq s,\\
&u(x,s) = u_s(x),\ \frac{\partial}{\partial t}u(x,s) = v_s(x) && x \in \Omega,\ s \in \mathbb{R},
\end{aligned}$$

where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 3$, is a bounded domain. We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and of sub-critical growth.

The initial value problem generates an asymptotically compact evolution process U in the phase space $V := H_0^1(\Omega) \times L^2(\Omega)$. We prove that the evolution process can be represented as a sum $U = S + C$, where the family of operators S satisfies the smoothing property with respect to V and an auxiliary normed space W compactly embedded into V, and C is a family of contractions in the stronger space V. Our main result in [5] implies the existence of a pullback exponential attractor for the evolution process U. Previous results cannot be applied since the constructions of exponential attractors were developed for evolution processes or semigroups that are asymptotically compact in the weaker space W, i.e., under the assumption that the family C is a contraction in W (among others see [11], [12] and [13]). Moreover, the former existence results for pullback exponential attractors in [8] and [16] required the Hölder continuity in time of the evolution process, which is generally not satisfied by hyperbolic equations.

The outline of our paper is as follows. In Section 2 we recall the main result of [5] about the existence of pullback exponential attractors for asymptotically compact evolution processes. We discuss properties of the pullback exponential attractors and consequences of our existence theorem in Section 3. Finally, in Section 4 we apply our theoretical results and show the existence of pullback exponential attractors for a non-autonomous damped wave equation and a non-autonomous Chafee-Infante equation.

2. **A general existence theorem for pullback exponential attractors.**

In this section we recall the existence result for pullback exponential attractors obtained in [5]. Let $U = \{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space $(V, \|\cdot\|_V)$. The construction of the pullback exponential attractor is based on the existence of a time-dependent pullback absorbing family, the compact embedding of the phase space into an auxiliary normed space and the asymptotic smoothing property of the process. We assume the process U can be represented as $U = S+C$, where $\{S(t,s)|\ t \geq s\}$ and $\{C(t,s)|\ t \geq s\}$ are families of operators satisfying the following properties:

($\mathcal{H}_0$) Let $(W, \|\cdot\|_W)$ be another normed space such that the embedding $V \hookrightarrow\hookrightarrow W$ is dense, compact and

$$\|v\|_W \leq \mu \|v\|_V \qquad \forall\, v \in V,$$

for some constant $\mu > 0$.

($\mathcal{H}_1$) There exists a family of bounded sets $B(t) \subset V$, $t \in \mathbb{T}$, that pullback absorbs all bounded subsets of V: For every bounded set $D \subset V$ and every $t \in \mathbb{T}$ there exists a pullback absorbing time $T_{D,t} \in \mathbb{T}_+ := \{t \in \mathbb{T} |\, t \geq 0\}$ such that

$$U(t, t-s)D \subset B(t) \qquad \forall\, s \geq T_{D,t}.$$

$(\mathcal{H}_2)$ The family $\{S(t,s)|\ t \geq s\}$ satisfies the smoothing property within the absorbing sets: There exist $\tilde{t} \in \mathbb{T}_+ \backslash \{0\}$ and a constant $\kappa > 0$ such that

$$\|S(t+\tilde{t},t)u - S(t+\tilde{t},t)v\|_V \leq \kappa \|u-v\|_W \qquad \forall\, u,v \in B(t),\ t \in \mathbb{T}.$$

$(\mathcal{H}_3)$ The family $\{C(t,s)|\ t \geq s\}$ is a contraction within the absorbing sets:

$$\|C(t+\tilde{t},t)u - C(t+\tilde{t},t)v\|_V \leq \lambda \|u-v\|_V \qquad \forall\, u,v \in B(t),\ t \in \mathbb{T},$$

where the contraction constant $0 \leq \lambda < \frac{1}{2}$.

$(\mathcal{H}_4)$ The process $\{U(t,s)|\ t \geq s\}$ is Lipschitz continuous within the absorbing sets: For all $t \in \mathbb{T}$ and $t \leq s \leq t+\tilde{t}$ there exists a constant $L_{t,s} > 0$ such that

$$\|U(s,t)u - U(s,t)v\|_V \leq L_{t,s} \|u-v\|_V \qquad \forall\, u,v \in B(t),\ t \in \mathbb{T}.$$

The construction of pullback exponential attractors requires to impose additional assumptions on the pullback absorbing family in Hypothesis $(\mathcal{H}_1)$.

(A_1) The family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ is positively semi-invariant for the evolution process $\{U(t,s)|\ t \geq s\}$,

$$U(t,s)B(s) \subset B(t) \qquad \forall\, t \geq s,\ t,s \in \mathbb{T}.$$

(A_2) For every bounded subset $D \subset V$ and time $t \in \mathbb{T}$ the corresponding absorbing times are bounded in the past: There exists $T_{D,t} \in \mathbb{T}_+$ such that

$$U(s,s-r)D \subset B(s) \qquad \forall\, s \leq t,\ r \geq T_{D,t}.$$

The above-stated assumptions allow to construct pullback exponential attractors for the evolution process $\{U(t,s)|\ t \geq s\}$ (see [5]).

Definition 2.1. We say that a time-dependent family of bounded sets $\{B(t)|\ t \in \mathbb{T}\}$ ***grows sub-exponentially in the past*** if

$$\operatorname{diam}(B(t))e^{\gamma t} \xrightarrow[t \to -\infty]{} 0 \qquad \forall\, \gamma > 0,$$

where $\operatorname{diam}(B)$ denotes the diameter of a subset $B \subset V$.

In the sequel, we denote by $B_r^X(a)$ the ball of radius $r > 0$ and center $a \in X$ in the metric space X and by $N_\varepsilon^X(A)$ the minimal number of balls in X with radius $\varepsilon > 0$ and centers in A needed to cover the subset $A \subset X$.

Theorem 2.2. *Let $\{U(t,s)|\ t \geq s\}$, $t,s \in \mathbb{T}$, be an evolution process in the Banach space V and the assumptions $(\mathcal{H}_0)$–$(\mathcal{H}_4)$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past. Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{T}\}$ for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}} \left(N_{\frac{\nu}{\kappa}}^W (B_1^V(0)) \right) \qquad \forall\, t \in \mathbb{T}.$$

Remark 1. For discrete evolution processes the Lipschitz continuity assumption $(\mathcal{H}_4)$ in Theorem 2.2 can be omitted.

3. **Properties of the pullback exponential attractor.** An immediate consequence of Theorem 2.2 is the existence and finite dimensionality of the global pullback attractor. For the proof of the following theorem we define the ***group of time shift operators*** or ***temporal translations*** $\{\mathcal{S}_r|\ r \in \mathbb{T}\}$ by

$$\mathcal{S}_r U(t,s) := U(t+r, s+r) \qquad t \geq s,\ t,s \in \mathbb{T},$$

where $r \in \mathbb{T}$ and $\{U(t,s)|\ t \geq s\}$ is an evolution process.

Theorem 3.1. *Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions $(\mathcal{H}_0)$–$(\mathcal{H}_3)$, (A_1) and (A_2) be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past. Then, the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{T}\}$ of the evolution process $\{U(t,s)|\ t \geq s\}$ exists, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{A}(t)) \leq \inf_{\nu \in (0, \frac{1}{2}-\lambda)} \left\{ \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right) \right\} \qquad \forall\, t \in \mathbb{T}.$$

Proof. For discrete evolution processes the statements follow from Theorem 2.2, Remark 1 and the minimality property of the global pullback attractor (see Definition 1.1).

Otherwise, if $\mathbb{T} = \mathbb{R}$, we define the discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$ by $\widetilde{U}(n,m) := U(n\tilde{t}, m\tilde{t})$ for all $n \geq m,\ n,m \in \mathbb{Z}$. It satisfies the assumptions of Theorem 2.2, and we conclude that for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu_d(k)|\ k \in \mathbb{Z}\}$ for the discrete evolution process $\{\widetilde{U}(n,m)|\ n \geq m\}$. We define the sets

$$\widehat{\mathcal{M}}^\nu(t) := U(t, k\tilde{t})\mathcal{M}^\nu_d(k) \qquad \text{for } t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z},$$

which implies $\widehat{\mathcal{M}}^\nu(k\tilde{t}) = \mathcal{M}^\nu_d(k)$ for all $k \in \mathbb{Z}$. Since the operators $U(t,s) : V \to V$, $t \geq s$, are continuous and the sections $\mathcal{M}^\nu_d(k)$, $k \in \mathbb{Z}$, are compact, $\{\widehat{\mathcal{M}}^\nu(t),|\ t \in \mathbb{R}\}$ is a family of compact subsets of V. Moreover, it follows as in the proof of Theorem 2.2 that the family $\{\widehat{\mathcal{M}}^\nu(t)|\ t \in \mathbb{R}\}$ pullback attracts all bounded subsets of V. By Theorem 1.2 we conclude that the global pullback attractor $\{\mathcal{A}(t)|\ t \in \mathbb{R}\}$ of the time continuous process $\{U(t,s)|\ t \geq s\}$ exists, and the minimality property implies $\mathcal{A}(t) \subset \widehat{\mathcal{M}}^\nu(t)$ for all $t \in \mathbb{R}$.

Since $\nu \in (0, \frac{1}{2} - \lambda)$ was arbitrary Theorem 2.2 implies that the fractal dimension of the discrete global pullback attractor is uniformly bounded by

$$\dim_{\mathrm{f}}^V(\mathcal{A}(k\tilde{t})) \leq \inf_{\nu \in (0, \frac{1}{2}-\lambda)} \left\{ \log_{\frac{1}{2(\nu+\lambda)}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right) \right\} \qquad \forall\, k \in \mathbb{Z},$$

and it remains to estimate the fractal dimension of the time continuous sections. To this end let $r \in \mathbb{R}$ be arbitrary. We consider the shifted evolution process $\{\mathcal{S}_r U(t,s)|\ t \geq s\}$ and the associated discrete evolution process $\{\widetilde{U}_r(n,m)|\ n \geq m\}$, which is given by $\widetilde{U}_r(n,m) := U_r(n\tilde{t}, m\tilde{t})$ for all $n \geq m,\ n,m \in \mathbb{Z}$. By Theorem 2.2 and Remark 1 for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu_{r,d}(k)|\ k \in \mathbb{Z}\}$ for the discrete evolution process $\{\widetilde{U}_r(n,m)|\ n \geq m\}$, and the fractal dimension of its sections satisfies the estimate stated in the theorem. We follow the previous arguments to conclude that the global pullback attractor $\{\mathcal{A}_r(t)|\ t \in \mathbb{R}\}$ for the time continuous evolution process $\{\mathcal{S}_r U(t,s)|\ t \geq s\}$ exists and observe that

$$\mathcal{A}_r(t) = \mathcal{A}(t+r) \qquad \forall\, t \in \mathbb{R}.$$

Moreover, the fractal dimension of the discrete sections of the global pullback attractor is uniformly bounded,

$$\begin{aligned}\dim_{\mathrm{f}}^{V}(\mathcal{A}_r(k\tilde{t})) &\leq \inf_{\nu\in(0,\frac{1}{2}-\lambda)}\left\{\dim_{\mathrm{f}}^{V}\left(\mathcal{M}^{\nu}_{r,d}(k)\right)\right\}\\ &\leq \inf_{\nu\in(0,\frac{1}{2}-\lambda)}\left\{\log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right)\right\}\end{aligned}$$

for all $k\in\mathbb{Z}$. Finally, since $r\in\mathbb{R}$ was arbitrary and

$$\mathcal{A}_r(k\tilde{t}) = \mathcal{A}(k\tilde{t}+r) \qquad \forall\, k\in\mathbb{Z},$$

we obtain the uniform bound for the fractal dimension of the time continuous global pullback attractor $\{\mathcal{A}(t)|\ t\in\mathbb{R}\}$. □

Remark 2. We remark that the Lipschitz continuity $(\mathcal{H}_4)$, which is essential for the construction of the time continuous pullback exponential attractor, is not required to establish the existence of the global pullback attractor and to derive estimates on its fractal dimension (see the hypothesis in Theorem 3.1).

Remark 3. The (Kolmogorov-) ε-entropy of a pre-compact subset $A\subset X$ is defined as

$$\mathcal{H}_\varepsilon(A) = \log_2(N^X_\varepsilon(A))$$

and was introduced by Kolmogorov and Tihomirov in [15]. The order of growth of $\mathcal{H}_\varepsilon$ as ε tends to zero is a measure for the massiveness of the set A in X, even if the fractal dimension of A is infinite.

The bound on the fractal dimension of the global pullback attractor in Theorem 3.1 is related to the entropy numbers for the embedding of the spaces V and W. For $k\in\mathbb{N}$ the k-th entropy number for the embedding $V\hookrightarrow W$ is defined as

$$e_k := \inf\left\{\varepsilon>0\,\Big|\, B^V_1(0)\subset\bigcup_{j=1}^{2^{k-1}} B^W_\varepsilon(w_j),\ w_j\in W, j=1,\ldots,2^{k-1}\right\}.$$

If V and W are infinite dimensional Banach spaces such that the embedding $V\hookrightarrow\hookrightarrow W$ is compact, then $0<e_k<\infty$ for all $k\in\mathbb{N}$.

Let $\lambda=0$ and $\nu\in(0,\frac{1}{2})$ be as in Theorem 3.1, that is, the evolution process U satisfies the smoothing property. Assuming that $e_k\to 0$ as $k\to\infty$ and that there exists $k\in\mathbb{N}$ such that $e_k=\frac{\nu}{\kappa}$ we obtain in our estimate

$$\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \leq \frac{(k-1)\ln(2)}{\ln(\frac{1}{2\kappa e_k})}.$$

We further observe that

$$\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \xrightarrow[\nu\to\frac{1}{2}]{} \infty.$$

On the other hand, if the entropy numbers grow polynomially in $\frac{1}{k}$, i.e., if $e_k=\frac{c}{k^\alpha}$ for some constants $c,\alpha>0$, then

$$\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \leq \frac{(k-1)\ln(2)}{\ln(\frac{1}{2\kappa c})+\alpha\ln(k)} \xrightarrow[k\to\infty]{} \infty,$$

and consequently,

$$\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \xrightarrow[\nu\to 0]{} \infty.$$

These observations illustrate that there exists an optimal constant $\nu \in (0, \frac{1}{2})$ to minimize the bound on the fractal dimension in Theorem 3.1.

For certain function spaces the entropy numbers can explicitly be estimated (see [10]). For instance, for the embeddings of the Sobolev spaces $W^{s_1,p}(\Omega)$ into $W^{s_2,q}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $s_1, s_2 \in \mathbb{R}$, $p, q \in (1,\infty)$ are such that $s_1 - s_2 - n \max\left\{0, \frac{1}{p} - \frac{1}{q}\right\} > 0$ it is known that

$$c_1 k^{-\frac{s_1-s_2}{n}} \leq e_k \leq c_2 k^{-\frac{s_1-s_2}{n}}$$

for some constants $c_1, c_2 \geq 0$ (Theorem 2, Section 3.3.3 in [10]), and our argumentation above applies.

The following proposition illustrates the relation between the global pullback and the pullback exponential attractor for evolution processes. We recall that an evolution process U was called $\widehat{B}$***-asymptotically compact*** in [19], where $\widehat{B} = \{B(t) | \, t \in \mathbb{T}\}$ is a family of bounded subsets, if for every $t \in \mathbb{T}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{T}_+$ and $(x_n)_{n\in\mathbb{N}}$ in $B(t-t_n)$ such that $\lim_{n\to\infty} t_n = \infty$ the set $\{U(t, t-t_n)x_n | \, n \in \mathbb{N}\}$ is pre-compact in V. Furthermore, the sets $\Lambda(\widehat{B}, t), t \in \mathbb{T}$, were defined as

$$\Lambda(\widehat{B}, t) := \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau)B(\tau)}^{\|\cdot\|_V} \qquad \forall\, t \in \mathbb{T},$$

where $\overline{A}^{\|\cdot\|_V}$ denotes the closure of the set A in V. Under the assumptions of Theorem 2.2 it can be observed from its proof in [5] that the evolution process U is $\widehat{B}$-asymptotically compact, where the family $\widehat{B}$ is the family of pullback absorbing sets $\widehat{B} = \{B(t) | \, t \in \mathbb{T}\}$ in Assumption $(\mathcal{H}_1)$. Moreover, it follows from [19] that $\Lambda(\widehat{B}, t), t \in \mathbb{T}$, is a strictly invariant family of non-empty, compact subsets of V that pullback attracts all bounded sets and

$$\mathcal{A}(t) \subset \Lambda(\widehat{B}, t) \qquad \forall\, t \in \mathbb{T}.$$

However, the sets do not coincide in general.

Remark 4. For an evolution process $\{U(t,s) | \, t \geq s\}$ satisfying the hypotheses of Theorem 2.2 the pullback exponential attractor in [5] was defined as

$$\mathcal{M}^\nu(t) = \overline{\bigcup_{n\in\mathbb{N}_0} U(t, k\tilde{t})E^n(k)}^{\|\cdot\|_V} \qquad \forall\, t \in [k\tilde{t}, (k+1)\tilde{t}[, \; k \in \mathbb{Z}$$

(see the proof of Theorem 3.2 and Theorem 3.3). The family of discrete sets $E^n(k), \; n \in \mathbb{N}_0, \, k \in \mathbb{Z}$, satisfies the properties

- $E^n(k) \subset U(k, k-n)B(k-n) \subset B(k)$,
- $\sharp E^n(k) \leq \sum_{l=0}^n N^l, \quad N := N^W_{\frac{\nu}{\kappa}}(B^V_1(0))$,
- $U(k, k-n)B(k-n) \subset \bigcup_{u\in E^n(k)} B^V_{(2(\nu+\lambda))^n R_{k-n}}(u)$,

where $\sharp$ denotes the cardinality of a set, $B(k)$ are the pullback absorbing sets for discrete times $k \in \mathbb{Z}$ in Hypothesis $(\mathcal{H}_1)$, and $R_k > 0$ is the radius of a ball in V that contains $B(k)$.

Proposition 1. *Let $\{U(t,s) | \, t \geq s\}$ be an evolution process in the Banach space V and the assumptions of Theorem 2.2 be satisfied. Then, the pullback exponential attractor of Theorem 2.2 can be represented as*

$$\mathcal{M}^\nu(t) = \Lambda(\widehat{B}, t) \cup \bigcup_{n\in\mathbb{N}_0} U(t, k\tilde{t})E^n(k) \qquad \forall\, t \in [k\tilde{t}, (k+1)\tilde{t}[, \; k \in \mathbb{Z},$$

where $\tilde t$ is given by $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, and we refer to [5] for the definition and construction of the family of sets $E^n(k)$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$.

Moreover, if the family of pullback absorbing sets is bounded in the past, i.e., if the union $\bigcup_{t\le t_0} B(t)$ is bounded for all $t_0\in\mathbb{T}$, then

$$\mathcal{M}^\nu(t)=\mathcal{A}(t)\cup\bigcup_{n\in\mathbb{N}_0} U(t,k\tilde t)E^n(k)\qquad \forall\, t\in[k\tilde t,(k+1)\tilde t[,\ k\in\mathbb{Z},$$

where $\{\mathcal{A}(t)|\ t\in\mathbb{T}\}$ denotes the global pullback attractor of the evolution process.

Proof. The pullback exponential attractor in [5] was defined as

$$\mathcal{M}^\nu(t)=\overline{\bigcup_{n\in\mathbb{N}_0} U(t,k\tilde t)E^n(k)}^{\|\cdot\|_V}\qquad \forall\, t\in[k\tilde t,(k+1)\tilde t[,\ k\in\mathbb{Z}.$$

Let $k\in\mathbb{Z}$, $t\in[k\tilde t,(k+1)\tilde t[$ and $x\in\mathcal{M}^\nu(t)$. Moreover, let $(x_m)_{m\in\mathbb{N}}$ be a sequence in $\bigcup_{n\in\mathbb{N}_0}U(t,k\tilde t)E^n(k)$ such that $\lim_{m\to\infty}x_m=x$. For every $m\in\mathbb{N}$ there exists $n_m\in\mathbb{N}_0$ such that $x_m\in U(t,k\tilde t)E^{n_m}(k)$. If $N_0:=\sup\{n_m|\, m\in\mathbb{N}\}<\infty$, then $\{x_m|\, m\in\mathbb{N}\}\subset\bigcup_{m=0}^{N_0}U(t,k\tilde t)E^{n_m}(k)$ and since the set is finite,

$$x=\lim_{m\to\infty}x_m\in\bigcup_{m=0}^{N_0}U(t,k\tilde t)E^{n_m}(k).$$

Otherwise, there exists a subsequence, which we denote by $(n_m)_{m\in\mathbb{N}}$ as well, such that $\lim_{m\to\infty}n_m=\infty$. By the definition of the sets $E^n(k)$ we have

$$x_m=U\big(t,(k-n_m)\tilde t\big)y_m$$

for some $y_m\in B\big((k-n_m)\tilde t\big)$. It follows that $x\in\Lambda(\widehat B,t)$, and we conclude

$$\mathcal{M}^\nu(t)\subset\Lambda(\widehat B,t)\cup\bigcup_{n\in\mathbb{N}_0}U(t,k\tilde t)E^n(k).$$

To show the reverse inclusion let $t\in\mathbb{T}$ and $x\in\Lambda(\widehat B,t)$. Then, there exist sequences $(t_m)_{m\in\mathbb{N}}$ in $\mathbb{T}_+$, $\lim_{m\to\infty}t_m=\infty$, and $(x_m)_{m\in\mathbb{N}}$ in $B(t-t_m)$ such that $x=\lim_{m\to\infty}U(t,t-t_m)x_m$. We argue by contradiction and assume that there exist $\varepsilon>0$ and $N_0\in\mathbb{N}$ such that

$$\operatorname{dist}_{\mathrm H}\big(U(t,t-t_m)x_m,\mathcal{M}^\nu(t)\big)\ge\varepsilon\qquad\forall m\ge N_0.$$

Let $k\in\mathbb{Z}$ be such that $t\in[k\tilde t,(k+1)\tilde t[$, and let $k_m\in\mathbb{Z}$, $s_m\in[0,\tilde t[$ be such that $t-t_m=(k-k_m)\tilde t-s_m$. We observe that

$$U\big((k-k_m)\tilde t,(k-k_m)\tilde t-s_m\big)x_m\in B\big((k-k_m)\tilde t\big)$$

and obtain by the definition of the pullback exponential attractor

$$\begin{aligned}&\operatorname{dist}_{\mathrm H}\Big(U(t,t-t_m)x_m,\mathcal{M}^\nu(t)\Big)\\ &\le\operatorname{dist}_{\mathrm H}\Big(U(t,k\tilde t)U(k\tilde t,t-t_m)x_m,U(t,k\tilde t)\bigcup_{n\in\mathbb{N}_0}E^n(k)\Big)\\ &\le L\operatorname{dist}_{\mathrm H}\Big(U(k\tilde t,t-t_m)x_m,\bigcup_{n\in\mathbb{N}_0}E^n(k)\Big)\\ &\le L\operatorname{dist}_{\mathrm H}\Big(U\big(k\tilde t,(k-k_m)\tilde t\big)B\big((k-k_m)\tilde t\big),\bigcup_{n\in\mathbb{N}_0}E^n(k)\Big),\end{aligned}$$

for some constant $L \geq 0$, where we used the Lipschitz-continuity ($\mathcal{H}_1$) in the second inequality and the semi-invariance of the absorbing sets in the last inequality. It follows from the proof of Theorem 2.2 in [5] that

$$U\big((k\tilde{t}, (k-k_m)\tilde{t})\big) B\big((k-k_m)\tilde{t}\big) \subset \bigcup_{u \in E^{k_m}(k)} B^V_{r_{k_m}}(u),$$

where the sequence of radii $r_{k_m} \to 0$ as k_m tends to ∞. We conclude that

$$\operatorname{dist}_{\mathrm{H}}\left(U(t, t-t_m)x_m, \mathcal{M}^\nu(t)\right) < \varepsilon$$

if $m \in \mathbb{N}$ is sufficiently large, which contradicts our assumption and shows the relation $\Lambda(\widehat{B}, t) \subset \mathcal{M}^\nu(t)$.

To prove the second statement in the proposition it suffices to show the inclusion

$$\mathcal{M}^\nu(t) \subset \mathcal{A}(t) \cup \bigcup_{n \in \mathbb{N}_0} U(t, k\tilde{t}) E^n(k) \qquad \forall\, t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z},$$

since the global pullback attractor is contained in the pullback exponential attractor. Let $k \in \mathbb{Z}$, $t \in [k\tilde{t}, (k+1)\tilde{t}[$, $x \in \mathcal{M}^\nu(t)$ and $(x_m)_{m\in\mathbb{N}}$ be a sequence in $\bigcup_{n\in\mathbb{N}_0} U(t, k\tilde{t})E^n(k)$ such that $\lim_{m\to\infty} x_m = x$. For every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}_0$ such that $x_m \in U(t, k\tilde{t})E^{n_m}(k)$. If $N_0 := \sup\{n_m |\, m \in \mathbb{N}\} < \infty$, it follows as above that

$$x = \lim_{m\to\infty} x_m \in \bigcup_{m=0}^{N_0} U(t, k\tilde{t}) E^{n_m}(k).$$

Otherwise, there exists a subsequence, which we denote by $(n_m)_{m\in\mathbb{N}}$ as well, such that $\lim_{m\to\infty} n_m = \infty$, and by the definition of the sets $E^n(k)$ we have

$$x_m = U\big(t, (k-n_m)\tilde{t}\big) y_m$$

for some $y_m \in B((k-n_m)\tilde{t})$. By assumption, the family of absorbing sets is bounded in the past, which implies that $\{y_m |\, m \in \mathbb{N}\} \subset \bigcup_{s\leq t} B(s) \subset D$ for some bounded set $D \subset V$. It follows that

$$x = \lim_{m\to\infty} x_m = \lim_{m\to\infty} U\big(t, (k-n_m)\tilde{t}\big) y_m \in \omega(D, t) \subset \mathcal{A}(t) = \overline{\bigcup_{\substack{D\,\subset\, X \\ \text{bounded}}} \omega(D,t)}^{\|\cdot\|_V},$$

where we used the representation of the global pullback attractor in Theorem 1.2. □

Remark 5. Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in V, the hypothesis of Theorem 2.2 be satisfied and $\{\mathcal{A}(t)|\, t \in \mathbb{T}\}$ and $\{\mathcal{M}^\nu(t)|\, t \in \mathbb{T}\}$ be the corresponding global and exponential pullback attractor. We remark that

$$\bigcup_{n\in\mathbb{N}_0} U(t, k\tilde{t}) E^n(k) \cap \mathcal{A}(t)$$

is a countable dense subset of the section $\mathcal{A}(t)$ of the global pullback attractor for every $t \in [k\tilde{t}, (k+1)\tilde{t}[,\ k \in \mathbb{Z}$.

Moreover, if the pullback exponential attractor is bounded in the past, Proposition 1 implies that the Hausdorff dimensions of the sections $\mathcal{A}(t)$ and of $\mathcal{M}^\nu(t)$ coincide,

$$\operatorname{dist}^V_{\mathrm{H}}(\mathcal{M}^\nu(t)) = \operatorname{dist}^V_{\mathrm{H}}(\mathcal{A}(t)) \qquad \forall t \in \mathbb{T},$$

since the Hausdorff dimension of every countable set is zero. In this case, if we required finite Hausdorff instead of finite fractal dimension in the definition of exponential attractors we could add an arbitrarily large countable semi-invariant set to the global attractor without changing its dimension. This is not possible if we impose finite fractal dimension in the definition of exponential attractors (see also [9], Chapter 7, for the autonomous case).

If an evolution process $\{U(t,s)|\ t\geq s\}$ possesses the global pullback attractor $\{\mathcal{A}(t)|\ t\in\mathbb{T}\}$ and is periodic, that is $\mathcal{S}_rU=U$ for some $r\in\mathbb{T}$, the invariance property

$$U(t,s)\mathcal{A}(s)=\mathcal{A}(t)\qquad \forall\, t\geq s,\ t,s\in\mathbb{T},$$

shows that the periodicity is directly inherited by the attractor. Since pullback exponential attractors are not unique we could certainly construct for an evolution process U and the shifted process $\mathcal{S}_rU$, where $r\in\mathbb{T}$, pullback exponential attractors $\mathcal{M}_U$ and $\mathcal{M}_{\mathcal{S}_rU}$ that do not satisfy the cocycle property

$$\mathcal{M}_U(t+r)=\mathcal{M}_{\mathcal{S}_rU}(t)\qquad \forall\, t,r\in\mathbb{T}.$$

However, if $\{\mathcal{M}_U(t)|\ t\in\mathbb{T}\}$ is a pullback exponential attractor for the evolution process U the translation of the attractor $\{\mathcal{M}_U(t+r)|\ t\in\mathbb{T}\}$ yields a pullback exponential attractor for the shifted process $\mathcal{S}_rU$, for every $r\in\mathbb{T}$.

Corollary 1. *Let $\{U(t,s)|\ t\geq s\}$ be an evolution process in the Banach space V. We assume that the hypotheses of Theorem 2.2 are satisfied and denote by $\{\mathcal{M}_U^\nu(t)|\ t\in\mathbb{T}\}$ the pullback exponential attractor for $\{U(t,s)|\ t\geq s\}$ in Theorem 2.2. Then, for every $r\in\mathbb{T}$ the family $\{\mathcal{M}_{\mathcal{S}_rU}^\nu(t)|\ t\in\mathbb{T}\}$, where*

$$\mathcal{M}_{\mathcal{S}_rU}^\nu(t):=\mathcal{M}_U^\nu(t+r)\qquad \forall\, t\in\mathbb{T},$$

is a pullback exponential attractor for the evolution process $\{\mathcal{S}_rU(t,s)|\ t\geq s\}$, and the family of exponential attractors satisfies

$$\mathcal{M}_U^\nu(t+r)=\mathcal{M}_{\mathcal{S}_rU}^\nu(t)\qquad \forall\, t,r\in\mathbb{T}.$$

If an evolution process is periodic the family of pullback exponential attractors $\{\mathcal{M}_{\mathcal{S}_rU}^\nu(t)|\ t\in\mathbb{T}\}_{r\in\mathbb{T}}$ exhibits the same property.

Proof. Let $r\in\mathbb{T}$, $\{\mathcal{M}_U^\nu(t)|\ t\in\mathbb{T}\}$ be the pullback exponential attractor for the evolution process $\{U(t,s)|\ t\geq s\}$ in Theorem 2.2 and

$$\mathcal{M}_{\mathcal{S}_rU}^\nu(t):=\mathcal{M}_U^\nu(t+r)\qquad\qquad \forall\, t\in\mathbb{T}.$$

Then, the family $\{\mathcal{M}_{\mathcal{S}_rU}^\nu(t)|\ t\in\mathbb{T}\}$ is semi-invariant under the action of the evolution process $\{\mathcal{S}_rU(t,s)|\ t\geq s\}$. The exponential pullback attraction property with respect to the process $\{\mathcal{S}_rU(t,s)|\ t\geq s\}$, the compactness of the sections and the uniform bound for its fractal dimension immediately follow from the corresponding properties of the family $\{\mathcal{M}_U^\nu(t)|\ t\in\mathbb{T}\}$, which proves that $\{\mathcal{M}_{\mathcal{S}_rU}^\nu(t)|\ t\in\mathbb{T}\}$ is a pullback exponential attractor for the shifted process. □

Finally, we formulate assumptions for the construction of forwards exponential attractors.

Definition 3.2. Let $\{U(t,s)|\ t\geq s\}$, $t,s\in\mathbb{T}$, be an evolution process in the metric space (X,d_X). We call the family $\mathcal{M}=\{\mathcal{M}(t)|\ t\in\mathbb{T}\}$ a ***forwards exponential attractor for the evolution process*** $\{U(t,s)|\ t\geq s\}$ if it satisfies Properties (i)-(iii) in Definition 1.3 and forwards exponentially attracts all bounded subsets of X: There exists a constant $\omega>0$ such that

$$\lim_{s\to\infty} e^{\omega s}\mathrm{dist}_{\mathrm{H}}\big(U(t+s,t)D,\mathcal{M}(t+s)\big)=0,$$

for every bounded subset $D\subset X$ and every $t\in\mathbb{T}$.

We replace the hypothesis $(\mathcal{H}_1)$ and (A_2) by the following:

$(\mathcal{H}_1)'$ There exists a family of bounded subsets $B(t) \subset V$, $t \in \mathbb{T}$, that forwards absorbs all bounded subsets of V: For every bounded set $D \subset V$ and every $t \in \mathbb{T}$ there exists a forwards absorbing time $T_{D,t} \in \mathbb{T}_+$ such that

$$U(t+s,t)D \subset B(t+s) \qquad \forall\, s \geq T_{D,t}.$$

$(A_2)'$ For every bounded subset $D \subset V$ and time $t \in \mathbb{T}$ the corresponding absorbing times are bounded in the future: There exists $T_{D,t} \in \mathbb{T}_+$ such that

$$U(s+r,s)D \subset B(s+r) \qquad \forall\, s \geq t,\ r \geq T_{D,t}.$$

Theorem 3.3. *Let $\{U(t,s)|\ t \geq s\}$ be an evolution process in the Banach space V and the assumptions $(\mathcal{H}_0)$, $(\mathcal{H}_1)'$, $(\mathcal{H}_2)$–$(\mathcal{H}_4)$, (A_1) and $(A_2)'$ be satisfied. Moreover, we assume that the diameter of the family of absorbing sets $\{B(t)|\ t \in \mathbb{T}\}$ grows at most sub-exponentially in the past.*

Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a forwards exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{T}\}$ for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B^V_1(0))\right) \qquad \forall t \in \mathbb{T}.$$

For discrete evolution processes Hypothesis $(\mathcal{H}_4)$ can be omitted.

Proof. Forwards exponential attractors can be constructed by slightly modifying the proof for pullback exponential attractors in [5]. □

Remark 6. If the pullback absorbing time $T_{D,t}$ corresponding to a bounded subset $D \subset X$ in Hypothesis $(\mathcal{H}_1)$ is independent of the time $t \in \mathbb{T}$, the family $\{B(t)|\ t \in \mathbb{T}\}$ is also forwards absorbing for the process. More precisely, the properties $(\mathcal{H}_1)$ and $(\mathcal{H}_1)'$ are indeed equivalent in this case, and the conditions (A_2) and $(A_2)'$ are automatically satisfied.

Consequently, in this case the pullback exponential attractor constructed in Theorem 2.2 coincides with the forwards exponential attractor in Theorem 3.3.

4. Applications.

In this section we illustrate our results and prove the existence of pullback exponential attractors for evolution processes generated by non-autonomous PDEs.

4.1. Non-autonomous Chafee-Infante equation.

The following initial value problem for the non-autonomous Chafee-Infante equation yields an example for a finite dimensional global pullback attractor which is unbounded in the past.

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with smooth boundary $\partial\Omega$ and $s \in \mathbb{R}$. We consider the initial-/boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \triangle u(x,t) + \lambda u(x,t) - \beta(t)\big(u(x,t)\big)^3 && x \in \Omega,\ t > s, \qquad (1)\\ \frac{\partial}{\partial \nu}u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s,\\ u(x,s) &= u_s(x) && x \in \Omega,\ s \in \mathbb{R}, \end{aligned}$$

where the constant $\lambda > 0$, Δ denotes the Laplace operator with respect to the spatial variable x, $\frac{\partial}{\partial \nu}$ the outward unit normal derivative on the boundary $\partial\Omega$ and $\frac{\partial}{\partial t}$ the partial derivative with respect to time $t > s$. The initial data u_s is a uniformly continuous function on Ω, $u_s \in C(\overline{\Omega})$. Moreover, we assume that the non-autonomous term $\beta : \mathbb{R} \to \mathbb{R}_+$ is strictly positive, continuously differentiable and satisfies the properties

$$0 < \sup_{t\in\mathbb{R}} \{\beta(t)\} \le \beta_0, \tag{2}$$

$$\lim_{t\to-\infty} \beta(t) = 0, \tag{3}$$

$$\sup_{t\in\mathbb{R}} \left\{ \frac{|\beta'(t)|}{\beta(t)} \right\} \le \beta_1, \tag{4}$$

$$\lim_{t\to-\infty} \frac{e^{\gamma t}}{\beta(t)} = 0 \qquad \forall\, \gamma > 0, \tag{5}$$

where the constants $0 < \beta_0, \beta_1 < \infty$. We consider the evolution process generated by (1) in the phase space $W := C(\overline{\Omega})$, where the norm in W is defined by

$$\|u\|_W := \max_{x\in\overline{\Omega}} |u(x)| \qquad u \in W.$$

To show the existence of a positively semi-invariant family of absorbing sets we use the method of lower and upper solutions (see [21], Chapter 2).

Definition 4.1. A function $u^* \in C(\overline{\Omega}\times[s,\infty[) \cap C^{2,1}(\Omega\times]s,\infty[)$ is called an ***upper solution*** for Problem (1) if it satisfies the inequalities

$$\begin{aligned} \frac{\partial}{\partial t}u^*(x,t) - \triangle u^*(x,t) &\ge \lambda u^*(x,t) - \beta(t)\big(u^*(x,t)\big)^3 && x \in \Omega,\ t > s, \\ \frac{\partial}{\partial \nu}u^*(x,t) &\ge 0 && x \in \partial\Omega,\ t \ge s, \\ u^*(x,s) &\ge u_s(x) && x \in \Omega,\ s \in \mathbb{R}. \end{aligned} \tag{6}$$

Analogously, the function $u_* \in C(\overline{\Omega}\times[s,\infty[) \cap C^{2,1}(\Omega\times]s,\infty[)$ is a ***lower solution*** for (1) if it satisfies the reversed inequalities in (6).

Lemma 4.2. *There exist constants $a, b \ge 0$ such that the function $c^* : [s,\infty[\to \mathbb{R}_+$,*

$$c^*(t) := \frac{a}{\sqrt{\beta(t)}} + b, \qquad t > s,$$

is an upper solution for (1) if the initial data satisfies $u_s(x) \le c^*(s)$ for all $x \in \Omega$.

If the initial function fulfils $u_s(x) \ge -c^*(s)$ for all $x \in \Omega$, the function $c_* : [s,\infty[\to \mathbb{R}$, $c_*(t) := -c^*(t)$, is a lower solution for (1).

Proof. Defining the function $c^*(t) := \frac{a}{\sqrt{\beta(t)}} + b$, where $a > \max\left\{\sqrt{\frac{\lambda}{3}}, \sqrt{\frac{\beta_1}{2} + \lambda}\right\}$ and $b > 0$ we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3 \\ &= \frac{a}{\sqrt{\beta(t)}} \left(-\frac{\beta'(t)}{2\beta(t)} + b\sqrt{\beta(t)}\Big(3a - \frac{\lambda}{a}\Big) + (a^2 - \lambda) + \frac{b^3}{a}\sqrt{\beta(t)}^3 + 3b^2\beta(t) \right). \end{aligned}$$

Since β vanishes slowly,

$$\sup_{t\in\mathbb{R}} \left\{ \frac{|\beta'(t)|}{\beta(t)} \right\} \le \beta_1 < \infty,$$

the choice of a and b implies

$$\frac{\partial}{\partial t}c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3 \ge 0 \qquad \forall t > s,$$

which proves that c^* is an upper solution for Problem (1).

The non-linearity is odd with respect to u, and hence, we obtain

$$\frac{\partial}{\partial t}c_*(t) - \triangle c_*(t) - \lambda c_*(t) + \beta(t)\big(c_*(t)\big)^3$$
$$= -\left(\frac{\partial}{\partial t}c^*(t) - \triangle c^*(t) - \lambda c^*(t) + \beta(t)\big(c^*(t)\big)^3\right) \leq 0.$$

Consequently, $c_* := -c^*$ is a lower solution for (1) if the initial data satisfies $u_s(x) \geq c_*(s)$ for all $x \in \Omega$. ☐

The linear heat equation

$$\frac{\partial}{\partial t}u(x,t) = \triangle u(x,t) \qquad x \in \Omega,\ t > 0, \tag{7}$$
$$\frac{\partial}{\partial \nu}u(x,t) = 0 \qquad x \in \partial\Omega,\ t \geq 0,$$
$$u(x,0) = u_0(x) \qquad x \in \Omega,$$

generates an analytic semigroup $\{e^{\Delta t} |\ t \in \mathbb{R}_+\}$ in the space $W := (C(\overline{\Omega}), \|\cdot\|_W)$ (see [20]). We denote the associated fractional power spaces by X^α, $\alpha \geq 0$. The operators $e^{\Delta t}$ are linear and bounded from W to X^α and satisfy the estimates

$$\|e^{\Delta t}\|_{\mathcal{L}(W;X^\alpha)} \leq \frac{C_\alpha}{t^\alpha} \qquad \forall\, t > 0, \tag{8}$$

where the constant $C_\alpha \geq 0$ and $\|\cdot\|_{\mathcal{L}(W;X^\alpha)}$ denotes the operator norm. The semi-linear problem (1) generates an evolution process $\{U(t,s) |\ t \geq s\}$ in W, where

$$U(t,s)u_s := u(\,\cdot\,, t; u_s, s) \qquad t \geq s,$$

and $u(\,\cdot\,,\,\cdot\,; u_s, s) : \overline{\Omega} \times [s,\infty[\to \mathbb{R}$ denotes the unique solution of (1) corresponding to initial data $u_s \in C(\overline{\Omega})$ and initial time $s \in \mathbb{R}$. Moreover, $\{U(t,s) |\ t \geq s\}$ satisfies the variation of constants formula

$$U(t,s)u_s = e^{\Delta(t-s)}u_s + \int_s^t e^{\Delta(t-\tau)} f(\tau, U(\tau,s)u_s))d\tau$$

(see [20] and [22]).

We apply Lemma 4.2 to show the existence of a semi-invariant family of pullback absorbing sets.

Proposition 2. *The family of subsets*

$$B(t) := \big\{v \in W \mid \|v\|_W \leq c^*(t)\big\}, \quad t \in \mathbb{R},$$

is positively semi-invariant for the evolution process $\{U(t,s) |\ t \geq s\}$ generated by Problem (1) and pullback absorbs all bounded sets of W.

Proof. Let $s \in \mathbb{R}$ and the initial data $u_s \in W$ satisfy $\|u_s\|_W \leq c^*(s)$. Lemma 4.2 implies that the functions c^* and c_* are upper and lower solutions for Problem (1). From Theorem 4.1, Chapter 2, in [21] it follows that there exists a unique classical solution $u(\,\cdot\,,\,\cdot\,; u_s, s) : \overline{\Omega} \times [s,\infty[\to \mathbb{R}$ and

$$c_*(t) \leq u(x,t; u_s, s) \leq c^*(t) \qquad \forall\, x \in \overline{\Omega}, t \geq s.$$

Consequently, the evolution process $\{U(t,s) |\ t \geq s\}$ satisfies

$$U(t,s)u_s \in B(t) \qquad \forall\, u_s \in B(s),\ t \geq s,$$

which proves the semi-invariance of the absorbing sets $\{B(t) |\ t \in \mathbb{R}\}$.

To show that the family is pullback absorbing, let $D \subset W$ be bounded and $t \in \mathbb{R}$. We choose $R > 0$ such that $D \subset B_R^W(0)$. By Assumption (3) there exists $t_0 \in \mathbb{R}$ such that $R \leq \frac{a}{\sqrt{\beta(t)}}$ for all $t \leq t_0$, and consequently, $D \subset B(t)$ for all $t \leq t_0$. Finally, we observe that the pullback absorbing time is bounded in the past, in particular, $T_{D,s} \leq t - t_0$ for all $s \leq t$. □

Next, we show that $\{U(t,s) | \ t \geq s\}$ satisfies the smoothing property with respect to the Banach spaces

$$V := \widehat{C}^1(\overline{\Omega}) := \Big\{ u \in C^1(\overline{\Omega}) \mid \frac{\partial}{\partial \nu} u(x) = 0, \, x \in \partial\Omega \Big\}$$

and W, where the norm in V is defined by

$$\|u\|_V := \|u\|_W + \sum_{j=1}^n \|\frac{\partial u}{\partial x_j}\|_W, \qquad u \in V.$$

Lemma 4.3. *Let $\{U(t,s) | \ t \geq s\}$ be the evolution process generated by Problem (1). Then, there exists a positive constant $\kappa > 0$ such that*

$$\|U(t+1,t)u - U(t+1,t)v\|_V \leq \kappa \|u - v\|_W \qquad \forall u, v \in B(t), \ t \in \mathbb{R}.$$

Proof. Let $s \in \mathbb{R}$ and $u, v \in B(s)$ be given initial data. We denote the corresponding solutions of Problem (1) by $u(t) := U(t,s)u$ and $v(t) := U(t,s)v$, $t \geq s$. It was shown in [20], Theorem 2.4, that the continuous embedding $X^\alpha \hookrightarrow V$ exists for all $\alpha > \frac{1}{2}$. Using the variation of constants formula we obtain

$$\begin{aligned} &\big\|u(t) - v(t)\big\|_V \leq \ c\big\|u(t) - v(t)\big\|_{X^\alpha} \\ &\leq c\Big(\big\|e^{\Delta(t-s)}(u-v)\big\|_{X^\alpha} + \int_s^t \big\|e^{\Delta(t-\tau)}\big(f(\tau,u(\tau)) - f(\tau,v(\tau))\big)\big\|_{X^\alpha} d\tau\Big) \\ &\leq c\big\|e^{\Delta(t-s)}\big\|_{\mathcal{L}(W;X^\alpha)}\big\|u-v\big\|_W \\ &\quad + c\int_s^t \big\|e^{\Delta(t-\tau)}\big\|_{\mathcal{L}(W;X^\alpha)}\big\|f(\tau,u(\tau)) - f(\tau,v(\tau))\big\|_W d\tau, \end{aligned}$$

where $c \geq 0$ denotes the embedding constant. By Proposition 2 it follows that

$$\begin{aligned} &\big\|f(\tau,u(\tau)) - f(\tau,v(\tau))\big\|_W \qquad (9) \\ &\leq \lambda\big\|u(\tau) - v(\tau)\big\|_W + \big\|\beta(\tau)\big(u(\tau) - v(\tau)\big)\big(u(\tau)^2 + u(\tau)v(\tau) + v(\tau)^2\big)\big\|_W \\ &\leq \lambda\big\|u(\tau) - v(\tau)\big\|_W + 2\big\|\big(u(\tau) - v(\tau)\big)\beta(\tau)\big(u(\tau)^2 + v(\tau)^2\big)\big\|_W \\ &\leq \lambda\big\|u(\tau) - v(\tau)\big\|_W + 4\big\|\big(u(\tau) - v(\tau)\big)\beta(\tau)\Big(\frac{a}{\sqrt{\beta(\tau)}} + b\Big)^2\big\|_W \\ &\leq (\lambda + C)\big\|u(\tau) - v(\tau)\big\|_W, \end{aligned}$$

for some constant $C \geq 0$, where we used Assumption (2) in the last estimate. The estimate (8) and the embedding $V \hookrightarrow W$ now imply

$$\begin{aligned} &\big\|u(t) - v(t)\big\|_V \\ &\leq cC_\alpha\Big(\frac{1}{(t-s)^\alpha}\big\|u - w\big\|_W + (\lambda + C)\int_s^t \frac{1}{(t-\tau)^\alpha}\big\|u(\tau) - v(\tau)\big\|_W d\tau\Big) \\ &\leq cC_\alpha\Big(\frac{1}{(t-s)^\alpha}\big\|u - w\big\|_W + (\lambda + C)\mu\int_s^t \frac{1}{(t-\tau)^\alpha}\big\|u(\tau) - v(\tau)\big\|_V d\tau\Big), \end{aligned}$$

for some constant $\mu > 0$. Finally, we set $t = s + 1$ and

$$y(s+1) := \|U(s+1,s)u - U(s+1,s)v\|_V = \|u(s+1) - v(s+1)\|_V,$$

and obtain the inequality

$$y(s+1) \le cC_\alpha\Big(\|u-v\|_W + (\lambda + C)\mu \int_s^{s+1} \frac{1}{(s+1-\tau)^\alpha} y(\tau)d\tau\Big).$$

Using the generalized Gronwall Lemma (Theorem 1.26 in [24]) we conclude

$$y(s+1) \le \kappa\|u-v\|_W,$$

for some constant $\kappa > 0$. □

Theorem 2.2 now implies the existence of a pullback exponential attractor in V for the evolution process $\{U(t,s)|\ t \ge s\}$.

Remark 7. For evolution processes that satisfy the smoothing property it suffices to assume that the pullback absorbing family is bounded in the metric of W and that the process satisfies the Lipschitz continuity property $(\mathcal{H}_4)$ in W.

Indeed, if the family of absorbing sets is bounded in the metric of W we define the sets

$$\widetilde{B}(t) := U(t, t-\tilde{t})B(t-\tilde{t}) \qquad t \in \mathbb{T},$$

which are pullback absorbing and bounded in the space V by the smoothing property $(\mathcal{H}_2)$. Moreover, the smoothing property $(\mathcal{H}_2)$, the Lipschitz continuity in W and the continuous embedding $(\mathcal{H}_0)$ imply

$$\begin{aligned}\|U(t+\tilde{t}+s,t)u - U(t+\tilde{t}+s,t)v\|_V &\le \kappa\|U(t+s,t)u - U(t+s,t)v\|_W \\ &\le \kappa L_{t,s}\|u-v\|_W \le \kappa L_{t,s}\mu\|u-v\|_V,\end{aligned}$$

for all $u, v \in B(t), t \in \mathbb{R}$ and $s \in [0,\tilde{t}]$. This proves the Lipschitz continuity of the evolution process in the space V and the result remains valid.

Theorem 4.4. *Let $\{U(t,s)|\ t \ge s\}$ be the evolution process in $W = C(\overline{\Omega})$ generated by Problem (1) and the function β satisfy Properties (2)–(5). Then, for every $\nu \in (0,\frac{1}{2})$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ in $V = \widehat{C}^1(\overline{\Omega})$ for the evolution process $\{U(t,s)|\ t \ge s\}$, and the fractal dimension of its sections is uniformly bounded by*

$$\dim_{\mathrm{f}}^V(\mathcal{M}^\nu(t)) \le \log_{\frac{1}{2\nu}}\Big(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\Big) \qquad \forall t \in \mathbb{R},$$

where $\kappa > 0$ denotes the smoothing constant in Lemma 4.3. Furthermore, the global pullback attractor exists and is unbounded in the past,

$$\lim_{t\to-\infty} \mathrm{diam}(\mathcal{A}(t)) \to \infty.$$

It is contained in the pullback exponential attractor, $\mathcal{A}(t) \subset \mathcal{M}^\nu(t)$, and

$$\dim_{\mathrm{f}}^V(\mathcal{A}(t)) \le \inf_{\nu\in(0,\frac{1}{2})}\Big\{\log_{\frac{1}{2\nu}}\Big(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\Big)\Big\} \quad \forall\, t \in \mathbb{R}.$$

Proof. The family of pullback absorbing sets $\{B(t)|\ t \in \mathbb{R}\}$ defined in Lemma 4.3 satisfies the hypothesis (A_1) and (A_2). Since the diameter of the absorbing sets is bounded by

$$\|B(t)\|_W \le 2\Big(\frac{a}{\sqrt{\beta(t)}} + b\Big) \qquad t \in \mathbb{R},$$

and the non-autonomous term satisfies Property (5), the absorbing sets grow at most sub-exponentially in the past. Moreover, the embedding $V \hookrightarrow\hookrightarrow W$ is compact, and the smoothing property with respect to the spaces V and W was shown in Lemma 4.3. To deduce the existence of a pullback exponential attractor from Theorem 2.2 it remains to verify the Lipschitz continuity of the evolution process. Let $s \in \mathbb{R}$ and $u, v \in B(s)$ be given initial data. Using the variation of constants formula we obtain

$$\begin{aligned}
&\big\|U(t,s)u - U(t,s)v\big\|_W \\
&\leq \big\|e^{\Delta(t-s)}(u-v)\big\|_W + \int_s^t \big\|e^{\Delta(t-\tau)}\big(f(\tau,U(\tau,s)u) - f(\tau,U(\tau,s)v)\big)\big\|_W d\tau \\
&\leq C_0\big\|u-v\big\|_W + C_0\int_s^t \big\|f(\tau,U(\tau,s)u) - f(\tau,U(\tau,s)v)\big\|_W d\tau \\
&\leq C_0\big\|u-v\big\|_W + C_0(\lambda + C)\int_s^t \big\|U(\tau,s)u - U(\tau,s)v\big\|_W d\tau,
\end{aligned}$$

for some constant $C_0 \geq 0$, where we used the estimate (9) in the proof of Lemma 4.3. By Gronwalls Lemma follows the Lipschitz continuity of $\{U(t,s)|\, t \geq s\}$ in W.

The global pullback attractor exists by Theorem 2.2, it is contained in the pullback exponential attractor and its sections are finite dimensional. The bound on the fractal dimension follows from Theorem 3.1, and it remains to show that the global pullback attractor is unbounded in the past. Due to the homogeneous Neumann boundary conditions, solutions of the ODE

$$\begin{aligned}
\frac{d}{dt}y(t) &= \lambda y(t) - \beta(t)(y(t))^3 && t > s, \\
y(s) &= y_0 && s \in \mathbb{R},\ y_0 \in \mathbb{R},
\end{aligned} \tag{10}$$

also solve Problem (1) with initial data $u_s(x) = y_0,\ x \in \overline{\Omega}$. As shown in [17], Proposition 3.1, for initial data $y_0 \neq 0$ the explicit solution of (10) is given by

$$y(t;s,y_0)^2 = \frac{e^{2\lambda t}}{e^{2\lambda s}y_0^{-2} + 2\int_s^t e^{2\lambda\tau}\beta(\tau)d\tau}, \qquad t > s.$$

Taking the limit $s \to -\infty$ we obtain two complete trajectories $\pm\xi$, where

$$\xi^2(t) = \frac{e^{2\lambda t}}{2\int_{-\infty}^t e^{2\lambda\tau}\beta(\tau)d\tau}, \qquad t \in \mathbb{R},$$

that are unbounded when t tends to $-\infty$ by Assumption (5).

If $\zeta(t), t \in \mathbb{R}$, is a complete trajectory of (10) above of $\xi(t), t \in \mathbb{R}$, the explicit solution formula implies

$$\zeta(t)^2 = \frac{e^{2\lambda t}}{e^{2\lambda s}\zeta(s)^{-2} + 2\int_s^t e^{2\lambda\tau}\beta(\tau)d\tau} > \xi(t)^2, \qquad t > s.$$

It follows that

$$\zeta(t)^2 > \frac{e^{2\lambda t}}{2\int_{-\infty}^t e^{2\lambda\tau}\beta(\tau)d\tau} = \xi(t)^2, \qquad t \in \mathbb{R},$$

which shows that solutions starting above of the complete trajectory ξ blow-up backwards in finite time and cannot be emanating from a bounded subset of $\mathbb{R}$.

We observe that $y(t) = 0,\ t \in \mathbb{R}$, is an equilibrium solution of (10), $\xi(t)$ pullback attracts at time t all solutions emanating from initial data $y_0 > 0$ and $-\xi(t)$ all solutions emanating from $y_0 < 0$. Moreover, the family of compact subsets

$$\{[-\xi(t), \xi(t)] \mid t \in \mathbb{R}\}$$

is strictly invariant for the evolution process generated by (10). By the connectedness of its sections it follows that the global pullback attractor $\mathcal{A}_{ode}$ of the ODE (10) is given by

$$\mathcal{A}_{ode}(t) = [-\xi(t), \xi(t)], \quad t \in \mathbb{R}.$$

When restricted to the subspace of constant functions, the evolution process $\{U(t,s) | t \geq s\}$ generated by Problem (1) coincides with the evolution process generated by the ODE (10), which implies that

$$\big\{u(\cdot,t) \in C(\overline{\Omega}) \mid u(x,t) = y(t)\ \forall\, x \in \overline{\Omega},\ y(t) \in [-\xi(t), \xi(t)]\big\} \subset \mathcal{A}(t), \quad t \in \mathbb{R},$$

and concludes the proof of the theorem. □

4.2. **Non-autonomous damped wave equation.** We consider the following initial value problem for the non-autonomous damped wave equation,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) + \beta(t)\frac{\partial}{\partial t}u(x,t) &= \Delta u(x,t) + f(u(x,t)) && x \in \Omega,\ t > s, \\
u(x,t) &= 0 && x \in \partial\Omega,\ t \geq s, \\
u(x,s) &= u_s(x) && x \in \Omega,\ s \in \mathbb{R}, \\
\frac{\partial}{\partial t}u(x,s) &= v_s(x) && x \in \Omega,\ s \in \mathbb{R},
\end{aligned} \tag{11}$$

where $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$. We assume that the non-linearity $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$|f'(z)| \leq c(1 + |z|^p), \qquad z \in \mathbb{R}, \tag{12}$$

$$\limsup_{|z|\to\infty} \frac{f(z)}{z} \leq 0, \tag{13}$$

for some constant $c > 0$ and $0 < p < \frac{2}{n-2}$. Furthermore, the function $\beta : \mathbb{R} \to \mathbb{R}_+$ is Hölder continuous and bounded from above and below by positive constants $0 < \beta_0 \leq \beta_1 < \infty$,

$$\beta_0 \leq \beta(t) \leq \beta_1 \qquad \forall\, t \in \mathbb{R}. \tag{14}$$

We apply Theorem 2.2 to show that the evolution process generated by (11) possesses a pullback exponential attractor. Setting $v := \frac{\partial}{\partial t}u$ and $w := \begin{pmatrix} u \\ v \end{pmatrix}$ we rewrite Problem (11) in the abstract form

$$\begin{aligned}
\frac{\partial}{\partial t}w &= A_\beta(t)w + F(w) && t > s, \\
w|_{t=s} &= w_s && w_s \in V, s \in \mathbb{R},
\end{aligned} \tag{15}$$

where the initial data $w_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix}$, and the phase space is $V := H_0^1(\Omega) \times L^2(\Omega)$. The norm in V is given by

$$\|w\|_V := \big(\|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}, \qquad w = \begin{pmatrix} u \\ v \end{pmatrix} \in V.$$

Furthermore, the operators are defined by $A_\beta(t) = A_1 + A_2(t)$,

$$A_1 := \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix}, \qquad A_2(t) := \begin{pmatrix} 0 & 0 \\ 0 & -\beta(t)Id \end{pmatrix}, \qquad F(w) := \begin{pmatrix} 0 \\ \widetilde{F}(u) \end{pmatrix},$$

where $A = -\Delta$ denotes the Laplace operator with homogeneous Dirichlet boundary conditions and domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ in $L^2(\Omega)$. The domain of the operator A_1 in V is $\mathcal{D}(A_1) = (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$, and $\widetilde{F}$ denotes the Nemytskii operator

$$\widetilde{F} : H_0^1(\Omega) \to L^2(\Omega), \qquad u \mapsto \widetilde{F}(u) := f(u(\cdot)).$$

The initial value problem (15) generates an evolution process $\{U(t,s)|\ t \geq s\}$ in the Banach space V, which is asymptotically compact and pullback strongly bounded dissipative. For details we refer to [4], Chapter 4 in [14], Section VI.4 in [6], [2] and [3].

We denote the evolution process generated by the linear homogeneous problem

$$\begin{aligned} \frac{\partial}{\partial t} w &= A_\beta(t) w & t &> s, \\ w|_{t=s} &= w_s & w_s &\in V, s \in \mathbb{R}, \end{aligned} \tag{16}$$

by $\{C(t,s)|\ t \geq s\}$. The following lemma was proved in [3] and yields the exponential decay of solutions of the linear homogeneous equation.

Lemma 4.5. *Let $\{C(t,s)|\ t \geq s\}$ be the evolution process in V associated to the linear problem (16). Then, there exist constants $C \geq 0$ and $\omega > 0$ such that the norm of the operators is bounded by*

$$\|C(t,s)\|_{\mathcal{L}(V;V)} \leq Ce^{-\omega(t-s)} \qquad \forall t \geq s,\ t,s \in \mathbb{R}.$$

The process $\{U(t,s)|\ t \geq s\}$ satisfies the integral equation

$$\begin{aligned} U(t,s)w_s &= C(t,s)w_s + \int_s^t C(t,\tau)F(U(\tau,s)w_s)d\tau \\ &= C(t,s)w_s + S(t,s)w_s \end{aligned}$$

(see [3] and [14]). Moreover, $\{U(t,s)|\ t \geq s\}$ is pullback strongly bounded dissipative and the pullback absorbing time corresponding to a bounded subset is independent of the time instant $t \in \mathbb{R}$. For the proof of the following lemma we refer to [3].

Lemma 4.6. *Let $\{U(t,s)|\ t \geq s\}$ be the evolution process in V generated by the initial value problem (15). Then, there exists a bounded uniformly pullback absorbing subset $B \subset V$, i.e., for every bounded set $D \subset V$ there exists $T_D \geq 0$ such that*

$$U(t,t-s)D \subset B \qquad \forall s \geq T_D,\ t \in \mathbb{R}.$$

To show that the family of operators $\{S(t,s)|\ t \geq s\}$ satisfies the smoothing property we establish several auxiliary results. We denote by X^α, $\alpha \in \mathbb{R}$, the fractional power spaces associated to the operator A with domain $\mathcal{D}(A) = X^1 = H_0^1(\Omega) \cap H^2(\Omega)$ in $X := L^2(\Omega)$ (see [23] or [22]). Furthermore, let $H^s(\Omega)$, $s \in \mathbb{R}_+$, be the fractional Sobolev spaces obtained by interpolation between the spaces $H^m(\Omega)$ and $L^2(\Omega)$, $m \in \mathbb{N}$ (see [1] or Section II.1.3 in [23]). Since the domain Ω is bounded we have the following continuous embeddings

$$H_0^s(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow L^2(\Omega) \qquad \text{if } \frac{1}{2} \geq \frac{1}{p'} \geq \frac{1}{2} - \frac{s}{n} > 0, \tag{17}$$

where $H_0^s(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ (see [1] or Theorem 1.1, Chapter 2, in [6]). If $\frac{1}{2} \geq \frac{1}{p'} > \frac{1}{2} - \frac{s}{n} > 0$ the embedding $H^s(\Omega) \hookrightarrow L^{p'}(\Omega)$ is compact. Moreover, Theorem 16.1 in [24] implies the continuous embeddings

$$H_0^s(\Omega) \hookrightarrow X^{\frac{s}{2}} \hookrightarrow H^s(\Omega) \qquad \forall s \in \mathbb{R}.$$

By duality we conclude

$$L^2(\Omega) \hookrightarrow L^{q'}(\Omega) \hookrightarrow X^{-\frac{s}{2}} \qquad \text{if } \frac{1}{p'} + \frac{1}{q'} = 1,\ \frac{1}{2} \geq \frac{1}{p'} \geq \frac{1}{2} - \frac{s}{n} > 0, \tag{18}$$

and the embedding $L^{q'}(\Omega) \hookrightarrow X^{-\frac{s}{2}}(\Omega)$ is compact if $\frac{1}{2} \geq \frac{1}{p'} > \frac{1}{2} - \frac{s}{n} > 0$.

The solution theory of the linear homogeneous problem can be extended to the fractional power spaces $X^\alpha \times X^{\alpha-\frac{1}{2}}$, $\alpha \in (0, \frac{1}{2})$ (see [23], Section IV.1.1).

Lemma 4.7. *Let $0 < \varepsilon < 1$ and $V^\varepsilon := X^{\frac{1-\varepsilon}{2}} \times X^{-\frac{\varepsilon}{2}}$. Then, for every initial data $w_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix} \in V^\varepsilon$, $s \in \mathbb{R}$, there exists a unique solution $w \in C([s, s+T]; V^\varepsilon)$ of the linear problem*

$$\begin{aligned} &\frac{\partial}{\partial t} w = A_\beta(t) w && s + T > t > s, \\ &w|_{t=s} = w_s && w_s \in V^\varepsilon, s \in \mathbb{R}, \end{aligned}$$

where $T > 0$ is arbitrary. Moreover, the generated evolution process is uniformly bounded in the space V^ε,

$$\|C(t,s)\|_{\mathcal{L}(V^\varepsilon; V^\varepsilon)} < d \qquad \forall\, t \geq s,\ t, s \in \mathbb{R},$$

for some constant $d \geq 0$.

Proof. We consider the operator

$$A_\beta(t) = A_1 + A_2(t) = \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\beta(t) Id \end{pmatrix},$$

in V^ε, where the operators $A_2(t) : V^\varepsilon \to V^\varepsilon$ are linear and uniformly bounded in t by Assumption (14), and A is considered as an operator in $X^{-\frac{\varepsilon}{2}}$ with domain $\mathcal{D}(A) = X^{1-\frac{\varepsilon}{2}}$. Since A is self-adjoint, the operator A_1 is dissipative in V^ε. Indeed, let $w = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A_1) = \mathcal{D}(A) \times X^{\frac{1-\varepsilon}{2}}$, then

$$\begin{aligned} \langle w, A_1 w \rangle_{V^\varepsilon} &= \langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} v \\ -Au \end{pmatrix} \rangle_{V^\varepsilon} = \langle A^{\frac{1-\varepsilon}{2}} u, A^{\frac{1-\varepsilon}{2}} v \rangle_X + \langle A^{-\frac{\varepsilon}{2}} v, A^{-\frac{\varepsilon}{2}}(-Au) \rangle_X \\ &= \langle A^{\frac{1-\varepsilon}{2}} u, A^{\frac{1-\varepsilon}{2}} v \rangle_X - \langle A^{\frac{1-\varepsilon}{2}} v, A^{\frac{1-\varepsilon}{2}} u \rangle_X = 0. \end{aligned}$$

By Corollary 4.4, Chapter 1, in [22] the operator A_1 generates a strongly continuous semigroup of contractions in V^ε. The lemma now follows from Theorem 1.2, Chapter 6, in [22]. ☐

Lemma 4.8. *There exists $0 < \varepsilon < 1$ such that the Nemytskii operator $\widetilde{F}$ is uniformly Lipschitz continuous from $H^{1-\varepsilon}(\Omega)$ to $L^2(\Omega)$ within bounded subsets of $H_0^1(\Omega)$: Let D be a bounded subset of $H_0^1(\Omega)$, then*

$$\|\widetilde{F}(u) - \widetilde{F}(v)\|_{L^2(\Omega)} \leq c\|u - v\|_{H^{1-\varepsilon}(\Omega)} \qquad \forall\, u, v \in D,$$

for some constant $c \geq 0$.

Proof. Let D be a bounded subset of $H_0^1(\Omega)$, $u, v \in D$ and $R > 0$ such that $D \subset B_R$, where $B_R := B_R^{H_0^1(\Omega)}(0)$. The assumption $p < \frac{2}{n-2}$ implies $p = (1-\varepsilon)\frac{2}{n-2}$ for some $0 < \varepsilon < 1$. Using the growth restriction (12) and Hölder's inequality with $p' = \frac{n}{2-2\varepsilon}$ and $q' = \frac{n}{n-2+2\varepsilon}$ we obtain

$$\begin{aligned}\|F(u) - F(v)\|_{L^2(\Omega)} &\le c\|(1 + |\zeta|^p)(u - v)\|_{L^2(\Omega)} \\ &\le c\big(\|u - v\|_{L^2(\Omega)} + \||\zeta|^p\|_{L^{2p'}(\Omega)}\|u - v\|_{L^{2q'}(\Omega)}\big) \\ &\le c\big(c_1\|u - v\|_{H^{1-\varepsilon}(\Omega)} + c_2\|\zeta\|^p_{L^{2pp'}(\Omega)}\|u - v\|_{H^{1-\varepsilon}(\Omega)}\big),\end{aligned}$$

for some $\zeta \in B_R$. In this estimate we used the continuous embeddings $H^{1-\varepsilon}(\Omega) \hookrightarrow L^2(\Omega)$ and $H^{1-\varepsilon}(\Omega) \hookrightarrow L^{2q'}(\Omega)$ in (17), and $c_1, c_2 \ge 0$ are the corresponding embedding constants. Since the set $D \subset B_R \subset H_0^1(\Omega)$ is bounded, the embedding $H_0^1(\Omega) \hookrightarrow L^{2pp'}(\Omega) = L^{\frac{2n}{n-2}}(\Omega)$ in (17) yields the uniform bound on the norm $\|\zeta\|^p_{L^{2pp'}(\Omega)}$ and concludes the proof of the lemma. □

Next, we show that the evolution process $\{U(t,s) |\ t \ge s\}$ restricted to the bounded pullback absorbing set B is uniformly Lipschitz continuous in the space $V^\varepsilon = X^{\frac{1-\varepsilon}{2}} \times X^{-\frac{\varepsilon}{2}}$, where $\varepsilon = 1 - \frac{p}{2}(n-2)$ was defined in the proof of Lemma 4.8.

Lemma 4.9. *Let $\varepsilon := 1 - \frac{p}{2}(n-2)$ and the initial data $w_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix} \in B$, $s \in \mathbb{R}$, where $B \subset V$ denotes the uniformly pullback absorbing set in Lemma 4.6. Then, the evolution process $\{U(t,s) |\ t \ge s\}$ generated by the initial value problem (15) is Lipschitz continuous in V^ε.*

Proof. By Lemma 4.8 the Nemytskii operator $\widetilde{F}$ is uniformly Lipschitz continuous from $H^{1-\varepsilon}(\Omega)$ to $L^2(\Omega)$ in bounded subsets of $H_0^1(\Omega)$. Let $B \subset H_0^1(\Omega)$ be bounded and $u, v \in B$. Using the continuous embeddings $L^2(\Omega) = X \hookrightarrow X^{-\frac{\varepsilon}{2}}$ and $X^{\frac{1-\varepsilon}{2}} \hookrightarrow H^{1-\varepsilon}(\Omega)$ we obtain

$$\begin{aligned}\|\widetilde{F}(u) - \widetilde{F}(v)\|_{X^{-\frac{\varepsilon}{2}}} &\le c_1\|\widetilde{F}(u) - \widetilde{F}(v)\|_X \\ &\le cc_1\|u - v\|_{H^{1-\varepsilon}(\Omega)} \le c_2\|u - v\|_{X^{\frac{1-\varepsilon}{2}}},\end{aligned} \tag{19}$$

for some constants $c_1, c_2 \ge 0$, which shows that $\widetilde{F}$ is uniformly Lipschitz continuous from $X^{\frac{1-\varepsilon}{2}}$ to $X^{-\frac{\varepsilon}{2}}$ in bounded subsets of $H_0^1(\Omega)$.

Let now $w_s, z_s \in B, s \in \mathbb{R}$, be given initial data and $w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = U(t,s)w_s$ and $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = U(t,s)z_s$ be the corresponding solutions of the semi-linear problem (15). The evolution process $\{U(t,s) |\ t \ge s\}$ is bounded in V by Lemma

4.6, and using the variation of constants formula we obtain

$$\begin{aligned}\|w(t)-z(t)\|_{V^\varepsilon} &\le \|C(t,s)\|_{\mathcal{L}(V^\varepsilon;V^\varepsilon)}\|w_s-z_s\|_{V^\varepsilon}+\\ &\quad+\int_s^t \|C(t,\tau)\|_{\mathcal{L}(V^\varepsilon;V^\varepsilon)}\|F(U(\tau,s)w_s)-F(U(\tau,s)z_s)\|_{V^\varepsilon}d\tau\\ &\le d\Big(\|w_s-z_s\|_{V^\varepsilon}+\int_s^t \|\widetilde{F}(w_1(\tau))-\widetilde{F}(z_1(\tau))\|_{X^{-\frac{\varepsilon}{2}}}d\tau\Big)\\ &\le d\Big(\|w_s-z_s\|_{V^\varepsilon}+\int_s^t c_2\|w_1(\tau)-z_1(\tau)\|_{X^{\frac{1-\varepsilon}{2}}}d\tau\Big)\\ &\le d\Big(\|w_s-z_s\|_{V^\varepsilon}+\int_s^t c_2\|w(\tau)-z(\tau)\|_{V^\varepsilon}d\tau\Big),\end{aligned}$$

where we used the estimate (19). The Lipschitz continuity now follows by Gronwall's Lemma,

$$\|U(t,s)w_s-U(t,s)z_s\|_{V^\varepsilon} \;=\; \|w(t)-z(t)\|_{V^\varepsilon}\le d\|w_s-z_s\|_{V^\varepsilon}e^{dc_2(t-s)}. \quad (20)$$

□

Combining the previous results we prove the smoothing property with respect to the spaces $V=X^{\frac{1}{2}}\times X$ and $W:=V^\varepsilon=X^{\frac{1-\varepsilon}{2}}\times X^{-\frac{\varepsilon}{2}}$ for the family of operators $\{S(t,s)|\ t\ge s\}$.

Lemma 4.10. *Let $\varepsilon=1-\frac{p}{2}(n-2)$ and $W:=V^\varepsilon$. Then, the embedding $V\hookrightarrow\hookrightarrow W$ is compact, and for every $t_0>0$ there exists a positive constant $\kappa_{t_0}>0$ such that*

$$\|S(t+t_0,t)w-S(t+t_0,t)z\|_V\le\kappa_{t_0}\|w-z\|_W \qquad \forall w,z\in B,\ t\in\mathbb{R},$$

where B denotes the uniformly pullback absorbing set defined in Lemma 4.6.

Proof. Let $s\in\mathbb{R}$, $t_0>0$ and $w,z\in B$ be given initial data. We denote the corresponding solutions of (15) by $U(t,s)w=\begin{pmatrix}U_1(t,s)w\\U_2(t,s)w\end{pmatrix}$ and $U(t,s)z=\begin{pmatrix}U_1(t,s)z\\U_2(t,s)z\end{pmatrix}$, $t\ge s$. By Lemma 4.5 and Lemma 4.9 follows

$$\begin{aligned}&\|S(s+t_0,s)w-S(s+t_0,s)z\|_V\\ \le\ &\int_s^{s+t_0}\|C(s+t_0,\tau)\big(F(U(\tau,s)w)-F(U(\tau,s)z)\big)\|_V d\tau\\ \le\ &C\int_s^{s+t_0}e^{-\omega(s+t_0-\tau)}\|\widetilde{F}(U_1(\tau,s)w-\widetilde{F}(U_1(\tau,s)z)\|_X d\tau\\ \le\ &Cc_1\int_s^{s+t_0}\|U_1(\tau,s)w-U_1(\tau,s)z\|_{H^{1-\varepsilon}(\Omega)}d\tau\\ \le\ &c_2\int_s^{s+t_0}\|U_1(\tau,s)w-U_1(\tau,s)z\|_{X^{\frac{1-\varepsilon}{2}}}d\tau\le c_2\int_s^{s+t_0}\|U(\tau,s)w-U(\tau,s)z\|_{V^\varepsilon}d\tau\\ \le\ &c_2d\int_s^{s+t_0}e^{c_2(\tau-s)}\|w-z\|_{V^\varepsilon}d\tau\le\kappa_{t_0}\|w-z\|_W,\end{aligned}$$

for some constants $c_1,c_2\ge 0$ and $\kappa_{t_0}>0$. In this estimate we used the continuous embedding $X^{\frac{1-\varepsilon}{2}}\hookrightarrow H^{1-\varepsilon}(\Omega)$ and the Lipschitz continuity (20) of the process $\{U(t,s)|\ t\ge s\}$ in V^ε. The compactness of the embedding $V\hookrightarrow\hookrightarrow W$ follows by (18). □

Finally, we show the existence of a pullback exponential attractor.

Theorem 4.11. *We set $\varepsilon = 1 - \frac{p}{2}(n-2)$. Let $\{U(t,s)|\ t \geq s\}$ be the evolution process in the Hilbert space $V = H_0^1(\Omega) \times L^2(\Omega)$ generated by the initial value problem (15) and $W = X^{\frac{1-\varepsilon}{2}} \times X^{-\frac{\varepsilon}{2}}$. Moreover, for arbitrary $\lambda < \frac{1}{2}$ we define $t_0 := \frac{1}{\omega}\ln\frac{C}{\lambda}$, where $C \geq 0$ and $\omega > 0$ denote the constants in Lemma 4.5.*

Then, for every $\nu \in (0, \frac{1}{2} - \lambda)$ there exists a pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$, which is also a forwards exponential attractor for the evolution process $\{U(t,s)|\ t \geq s\}$, and the fractal dimension of its sections is uniformly bounded by

$$\dim_f^V(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right) \qquad \forall t \in \mathbb{R},$$

where $\kappa = \kappa_{t_0} > 0$ denotes the smoothing constant in Lemma 4.10. Furthermore, the global pullback attractor exists, is contained in the pullback exponential attractor $\{\mathcal{M}^\nu(t)|\ t \in \mathbb{R}\}$ and

$$\dim_f^V(\mathcal{A}(t)) \leq \inf_{\nu \in (0, \frac{1}{2}-\lambda)} \left\{\log_{\frac{1}{2(\nu+\lambda)}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right)\right\} \qquad \forall t \in \mathbb{R}.$$

Proof. By Lemma 4.6 there exists a fixed bounded uniformly pullback absorbing set $B \subset V$, and the pullback and forwards absorbing assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_1)'$, (A_1), (A_2) and $(A_2)'$ are satisfied. If $\lambda \in (0, \frac{1}{2})$ and $t_0 = \frac{1}{\omega}\ln\frac{C}{\lambda}$, Lemma 4.5 implies that the linear operators $C(t+t_0, t)$, $t \in \mathbb{R}$, are contractions in V with contraction constant $\lambda < \frac{1}{2}$, which verifies Hypothesis $(\mathcal{H}_3)$ with $\tilde{t} = t_0$. Moreover, we proved in Lemma 4.10 that the smoothing property $(\mathcal{H}_2)$ of the family of operators $\{S(t,s)|\ t \geq s\}$ is valid within the absorbing set B. To show the Lipschitz continuity $(\mathcal{H}_4)$ of the evolution process we recall that the Nemytskii operator $\widetilde{F}$ is locally Lipschitz continuous from $H^{1-\varepsilon}(\Omega)$ to $L^2(\Omega)$ (see Lemma 4.8). If the subset $D \subset H_0^1(\Omega)$ is bounded the continuous embedding $H_0^1(\Omega) \hookrightarrow H^{1-\varepsilon}(\Omega)$ implies

$$\|\widetilde{F}(u) - \widetilde{F}(v)\|_{L^2(\Omega)} \leq c\|u-v\|_{H^{1-\varepsilon}(\Omega)} \leq cc_1\|u-v\|_{H_0^1(\Omega)} \qquad \forall u, v \in D, \tag{21}$$

for some constant $c_1 \geq 0$. We can now show the Lipschitz continuity of the evolution process $\{U(t,s)|\ t \geq s\}$ in V as in the proof of Lemma 4.9 if we replace the space V^ε by V and use the estimate (21) and Lemma 4.5 instead of the estimate (19) and Lemma 4.7, respectively.

Consequently, all required hypothesis are verified and the existence of the pullback exponential attractor and the uniform estimates for the fractal dimension of its sections follow from Theorem 2.2. Theorem 3.3 implies that the pullback exponential attractor is also a forwards exponential attractor for the evolution process. Moreover, the global pullback attractor of the evolution process exists, is contained in the pullback exponential attractor, and Theorem 3.1 yields the bound for the fractal dimension of its sections, which concludes the proof. ☐

Acknowledgments. We thank the anonymous referee for his careful reading of the manuscript and his valuable comments and remarks that greatly improved the presentation of our results. The article was completed while the second author was visiting the *Instituto de Ciências Matemáticas e de Computação*, Universidade de São Paulo at São Carlos. She would like to express her gratitude to the colleagues of the institute for their warm hospitality and *FAPESP* and *BCAM* for financially supporting the visit.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd edition, Elsevier, Amsterdam, 2003.

[2] J. Arrieta, A. N. Carvalho and J. K. Hale, A damped hyperbolic equation with critical exponent, *Comm. Partial Differential Equations*, **17** (1992), 841–866.

[3] T. Caraballo, A. N. Carvalho, J. A. Langa and L. F. Rivero, Existence of pullback attractors for pullback asymptotically compact processes, *Nonlinear Anal.*, **72** (2010), 1967–1976.

[4] A. N. Carvalho, J. A. Langa and J. C. Robinson, *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, *Appl. Math. Sci.*, **182**, Springer, 2012.

[5] A. N. Carvalho and S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: theoretical results, *Commun. Pure and Appl. Anal.*, **12** (2013), 3047–3071.

[6] V. Chepyzhov and M. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc., Providence, RI, 2002.

[7] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dynam. Differential Equations*, **9** (1997), 307–341.

[8] R. Czaja and M. A. Efendiev, Pullback exponential attractors for nonautonomous equations part I: Semilinear parabolic equations, *J. Math. Anal. Appl.*, **381** (2011), 748–765.

[9] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Research in Applied Mathematics, Masson, Paris, John Wiley & Sons, Ltd., Chichester, 1994.

[10] D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers and Differential Operators*, Cambridge University Press, New York, 1996.

[11] M. A. Efendiev, A. Miranville and S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, *C. R. Acad. Sci. Paris Sr. I Math.*, **330** (2000), 713–718.

[12] M. A. Efendiev, A. Miranville and S. Zelik, Exponential attractors and finite-dimensional reduction for nonautonomous dynamical systems, *Proc. R. Soc. Edinburgh Sect. A*, **135A** (2005), 703–730.

[13] M. A. Efendiev, Y. Yamamoto and A. Yagi, Exponential attractors non-autonomous dissipative systems, *J. Math. Soc. Japan*, **63** (2011), 647–673.

[14] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, Providence, Rhode Island, 1988.

[15] A. N. Kolmogorov and V. M. Tihomirov, ε-entropy and ε-capacity of sets in functional spaces, *Amer. Math. Soc. Transl. Ser. 2*, **17** (1961), 277–364.

[16] J. A. Langa, A. Miranville and J. Real, Pullback exponential attractors, *Discrete Contin. Dyn. Syst.*, **26** (2010), 1329–1357.

[17] J. A. Langa, J. C. Robinson and A. Suárez, *Stability, instability and bifurcation phenomena in non-autonomous differential equations*, *Nonlinearity*, **15** (2002), 887–903.

[18] J. A. Langa and B. Schmalfuss, Finite dimensionality of attractors for non-autonomous dynamical systems given by partial differential equations, *Stoch. Dyn.*, **4** (2004), 385–404.

[19] P. Marín-Rubio and J. Real, On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Anal.*, **71** (2009), 3956–3963.

[20] X. Mora, Semilinear parabolic problems define semiflows on C^k spaces, *Trans. Amer. Math. Soc.*, **278** (1983), 21–55.

[21] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[22] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[23] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Appl. Math. Sci. 68, Springer Verlag, New York, 1997.

[24] A. Yagi, *Abstract Parabolic Evolution Equations and Their Applications*, Springer-Verlag, Berlin-Heidelberg, 2010.

Received April 2013; revised September 2013.

E-mail address: andcarva@icmc.usp.br
E-mail address: ssonner@bcamath.org, stefanie.sonner@helmholtz-muenchen.de