Wavelet Sets and Reflection Groups

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Garching, den 30. November 2007

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Abstract

In this diploma thesis the emerging theory of dilation-reflection wavelet sets is studied, by which a multiresolution analysis is contructed via a Hilbert space basis of fractal functions on foldable figures. Our work centers on the notion of *triple wavelet sets*, a type of sets that tesselate Euclidean space under three distinct transformation groups. We give existence proofs for such wavelet sets in all dimensions and provide examples. We found a link to the third Hilbert problem and apply its solution to our tesselation problems. Finally, we include a comparative study of our approach with some recent similar work by the research group of Prof. Guido Weiss in a designated section.

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Contents

Chapter 1

A Brief Introduction to Wavelets

Wavelet theory is a fairly new answer to a fairly old problem – the problem of finding convenient linear representations of functions belonging to a Hilbert space. It has been motivated in particular by the need to model signals that are transient, i.e. functions that decay fast or have compact support. Wavelets are a way of mending various shortcomings that classical Fourier analysis encounters in this field.

Approximation theory, of which wavelet theory is a subfield, is concerned with finding (generalised) bases for function spaces like $L^2(\mathbb{R})$. It is intuitively obvious that the elements of such a basis should resemble the functions we are trying to represent. Given a compactly supported signal, we will achieve better convergence when using compactly supported basis functions than if stationary functions were used.

A typical example of the drawbacks when using the latter method is the problem of the *Gibbs effect*, which produces "overshooting" oscillations in the approximation of functions with jump discontinuities (e.g. a square wave) even when using series up to high orders. In image compression, an instance of approximating two-dimensional signals, this effect is responsible for flaws in the rendering of images that contain edges of sharp contrast. Since signals with discontinuities are common in applications, the advantage of knowing basis functions localised in time (or space) is clear.

In this chapter we will first give a brief exposition of general wavelet theory, establishing common terminology and giving examples. In a second subsection, we will discuss wavelet sets, with some emphasis on the situation in higher dimensions. For general introductory reading, we refer to the book by Hernandez and Weiss [HW96] and Mallat's book [SM99].

1.1 General Principles

A *wavelet* in its widest sense is an element $\varphi(t)$ of a Hilbert space of functions $\mathcal{H}$ which generates a basis under a system of unitary operators that shift φ in both time and frequency. Possible variations in the definition depend on the underlying function space (e.g. $L^2(\mathbb{R}^n)$, with functions on manifolds as a possible generalisation), the unitary system acting on it (usually dilations and translations, but possibly also reflections, rotations or shears), the kind of basis generated (a complete orthonomal basis, a Riesz basis or a frame), and occasionally the measure on $\mathcal{H}$. As in Fourier analysis, there are discrete and continuous versions, but we will focus on the discrete case here. For the sake of exposition we give the most common and straightforward definition, and generalise gradually.

1.1 Definition. A *dyadic orthonormal wavelet in one dimension* (also called a *mother wavelet*) is a unit vector $\varphi \in L^2(\mathbb{R})$, such that the system

$$\left\{\varphi_{j,k}(t) := 2^{\frac{j}{2}}\varphi(2^j t - k) : j, k \in \mathbb{Z}\right\}$$

is a complete orthonormal basis of $L^2(\mathbb{R})$.

1.2 Example. An instance for which the above conditions are immediately verified is the well-known *Haar wavelet*, given by

$$\varphi(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t).$$

This is an example of a discontinuous wavelet; however there also exist wavelets of arbitrary degree of smoothness.

Returning to our definition, we can see that the above system can be rewritten as $\left\{D^jT^k\varphi : j, k \in \mathbb{Z}\right\}$, where D, T are unitary operators acting on $L^2(\mathbb{R})$ via

$$(D\varphi)(t) = \sqrt{2}\varphi(2t)\,, \qquad (T\varphi)(t) = \varphi(t-1).$$

In other words, these *dilation and translation operators* constitute a unitary system $\mathcal{U} = \{D^jT^k : j, k \in \mathbb{Z}\}$ (denoted $\mathcal{U}_{D,T}$ or $\langle D, T\rangle$ in [DL98]) which realises the shifts in frequency and time. This is where we will see some far-reaching generalisations in this treatise.

Proceeding with generalities, as an important basic result we mention at this point a widely used criterion for all dyadic orthonormal 1-dimensional wavelets.

1.3 Proposition. A unit vector $\varphi \in L^2(\mathbb{R})$ is a wavelet precisely if the following two conditions are satisfied:

$$\sum_{j\in\mathbb{Z}} |\hat{\varphi}(2^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R},$$

and for every odd integer m, we have

$$\sum_{j=0}^{\infty} \hat{\varphi}(2^j\xi)\overline{\hat{\varphi}(2^j(\xi + 2m\pi))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

For a proof, we refer to chapter 7 of Weiss's book [HW96].

Fourier transform. Throughout this work, we will use the normalised Fourier-Plancherel transform, given by

$$(\mathscr{F}f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} s^{-ist} f(t)dt = \widehat{f}(s)$$

for all $f \in L^2(\mathbb{R})$. This way, the Fourier transform becomes a unitary operator and hence an isometric automorphism of $L^2(\mathbb{R})$.

We will also need the concept of a *Multiresolution Analysis*, defined as follows.

1.4 Definition. A *Multiresolution Analysis (MRA)* on the space $L^2(\mathbb{R})$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$, satisfying the following properties:

1. $V_j \subset V_{j+1} \qquad$ for all $k \in \mathbb{Z}$;
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
3. $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$;
4. there exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

Multiresolution Analyses were introduced by Meyer and Mallat in 1989 as a framework for constructing discrete wavelet transforms. To interpret the above definition in the context of wavelets, note that by the above, an MRA consists mainly of a sequence of nested subspaces of $L^2(\mathbb{R})$, each V_j being a "refinement in frequency" of the previous space V_{j-1}. By defining the *wavelet spaces* W_j as the orthogonal complement of V_j in V_{j+1}, i.e. $W_j = V_{j+1} \ominus V_j$, the space $L^2(\mathbb{R})$ decomposes into a direct sum of orthogonal subspaces $\bigoplus_{j\in\mathbb{Z}} W_j$. The functions φ_j giving rise to a basis for each V_j under $\mathbb{Z}$-translation are known as the *scaling functions* or *father wavelets*. Moreover, the above definition implies that for each j, there exists a function ψ_j, i.e. the actual wavelet function or *mother wavelet*, whose $\mathbb{Z}$-translates form a basis of W_j. In short, these ψ_j are wavelet functions precisely in the sense of Definition 1.1. A proof and thorough discussion of these important facts can be found in [HW96], chapter 2.

Besides being ubiquitous in wavelet theory, we mention this concept here because in chapter 2 we will see, under a slight alteration of Definition 1.4, how this concept may be extended to a theory of dilation-reflection wavelets in an interesting way.

1.2 MSF Wavelets and Wavelet Sets

This section introduces a class of wavelets that will dominate the rest of this work.

1.5 Definition (Wavelet set)**.** A measurable set $E \subseteq \mathbb{R}$ is called a *wavelet set* if the scaled characteristic function $\frac{1}{\sqrt{\lambda(E)}}\chi_E$ is the Fourier transform of a wavelet. A wavelet that arises in this way is called a *minimally supported frequency (MSF) wavelet* (sometimes also called an *s-elementary wavelet*).

1.6 Remark. The motivation for the name "MSF" is that orthonormal wavelets have the property that $\lambda(\operatorname{supp}\widehat{\varphi}) \geq 1$ (where λ denotes Lebesgue measure), with equality if and only if $|\widehat{\varphi}| = \frac{1}{\sqrt{\lambda(E)}}\chi_E$ holds a.e. for some measurable E in $\mathbb{R}$.

Main Example. Let $S = [-2\pi, -\pi] \cup [\pi, 2\pi]$. Then the function φ_S, defined by $\widehat{\varphi_S} = \frac{1}{\sqrt{2\pi}}\chi_S$, is the so-called *Shannon wavelet*, also known as the *Littlewood-Paley wavelet*, displayed in the following graph. The set S is known as the *Shannon set.*

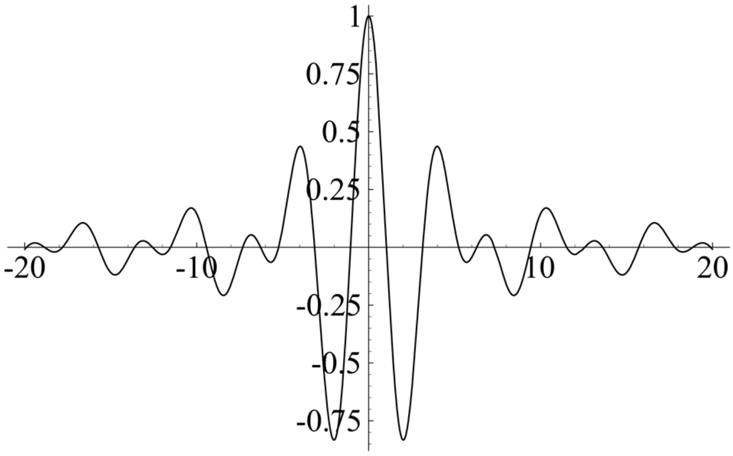


FIGURE 1.2.

1.7 Proposition. The Shannon wavelet is indeed a wavelet.

Proof. This could be shown by applying Proposition 1.3, but in fact it will also follow from Theorem 1.14, in which the Shannon set gives a point of departure for describing all other 1-dimensional wavelet sets. □

The next two sections will be devoted to a complete characterisation of wavelet sets, first in one dimension and then generalised to sets in $\mathbb{R}^n$. For this, let us first establish some terminology relating to group actions on $\mathbb{R}^n$.

1.8 Definition (G-tile)**.** Let $\mathcal{G}$ be a group acting on $\mathbb{R}^n$ which maps measurable sets to measurable sets. A measurable set $X \subseteq \mathbb{R}^n$ is called a *$\mathcal{G}$-tile* (or *$\mathcal{G}$-tiling set*, or *$\mathcal{G}$-fundamental domain*, or a *$\mathcal{G}$-generator of a partition of* $\mathbb{R}^n$) if $\mathbb{R}^n$ is a disjoint union of the $\mathcal{G}$-copies of X:

$$\mathbb{R}^n = \dot{\bigcup_{g\in\mathcal{G}}} g(X).$$

For pairs of groups $(\mathcal{G}_1, \mathcal{G}_2)$ (or analogously triples, as seen later), a $(\mathcal{G}_1, \mathcal{G}_2)$*-tile* is a set which is both a $\mathcal{G}_1$-tile and a $\mathcal{G}_2$-tile.

1.9 Remark. The term "fundamental domain" is usually only applied when the tile in question is a compact connected set (by contrary, this will not be assumed of a $\mathcal{G}$-tile in general).

1.10 Remark. The usual neglection of Lebesgue-null sets will be tacitly applied throughout (in particular, unless otherwise specified, we will use the term "disjoint" to mean "possibly containing Lebesgue-null sets in the intersection", and for a "measurable partition" of a set $X = \bigcup_i X_i$ (with X, X_i measurable), the set $X \setminus \bigcup_i X_i$ may be a Lebesgue-null set). Further, any subsets of $\mathbb{R}^n$ mentioned throughout this text will generally be assumed to be Lebesgue measurable.

1.11 Definition ($\mathcal{G}$-congruence)**.** Let $\mathcal{G}$ be a group of measurable transformations acting on $\mathbb{R}^n$, and $E, F \subseteq \mathbb{R}^n$ be measurable sets. Then E, F are said to be $\mathcal{G}$-*congruent* or *congruent modulo* $\mathcal{G}$ (in symbols, $E \sim_{\mathcal{G}} F$) if there exist measurable partitions $\{E_g : g \in \mathcal{G}\}$, $\{F_g : g \in \mathcal{G}\}$ of E and F such that $F_g = g(E_g)$ for all $g \in \mathcal{G}$.

1.12 Proposition. $\mathcal{G}$-congruence is an equivalence relation.

1.13 Proposition. If E is a $\mathcal{G}$-tile, then F is also a $\mathcal{G}$-tile iff $E \sim_{\mathcal{G}} F$.

The proofs are elementary.

1.2.1 Characterisation of 1-dimensional MSF wavelets

Wavelet sets in 1 dimension, and hence the associated MSF wavelets, are characterised in a very simple way by means of the group of dilations by powers of 2 and the group of translations by $2k\pi$ acting on $\mathbb{R}$.

1.14 Theorem. *A measurable set $E \subseteq \mathbb{R}$ is a wavelet set iff it is both dilation-2 congruent and translation-2π congruent to the Shannon wavelet set $S = [-2\pi, -\pi) \cup [\pi, 2\pi)$.*

Before the proof, let us briefly determine how the actions of D and T, i.e. dilation by 2 and translation by 1, behave under the Fourier transform. It is easily seen that

$$(\mathscr{F}T\varphi)(s) = \int_{\mathbb{R}} e^{-ist}\varphi(t-1)dt = e^{-is}(\mathscr{F}\varphi)(s),$$

whence, replacing φ by $\mathscr{F}^{-1}\varphi$, we have $(\mathscr{F}T\mathscr{F}^{-1})\varphi = e^{-is}\varphi$. Using M_h to denote the multiplication operator $f \mapsto hf$, and denoting by $\widehat{T}$ the "conjugation" $\mathscr{F}T\mathscr{F}^{-1}$ of T by the Fourier transform, we obtain that $\widehat{T} = M_{e^{-is}}$.
A similar computation for the dilation operator D verifies that $\widehat{D} = D^{-1}$. For (positive or negative) powers of these operators, we have

$$\widehat{T^n} = \mathscr{F}T^n\mathscr{F}^{-1} = (\mathscr{F}T\mathscr{F}^{-1})^n = M_{e^{-ins}}\ ,$$

and similarly,

$$\widehat{D^n} = D^{-n}.$$

After these observations we return to showing our result.

Proof. We start with the backwards implication. Let E be dilation-2 and translation-2π congruent to $S = [-2\pi, -\pi) \cup [\pi, 2\pi)$. It is well-known that that the set of normalised exponentials $\{\frac{1}{\sqrt{2\pi}} e^{iks} : n \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2([0, 2\pi])$. Since these exponentials are 2π-periodic, they will also constitute an ONB when defined on any set that is 2π-translation congruent to $[0, 2\pi]$ – this holds, in particular, for the Shannon set. Therefore, if E is 2π-translation congruent to S, we have found an orthonormal basis for $L^2(E)$.

Next, given our evaluation of the operator T in the Fourier domain, we can see that a complex exponential $\frac{1}{\sqrt{2\pi}} e^{iks}$, defined on E, may be rewritten as $M_{e^{iks}} \frac{1}{\sqrt{2\pi}} = \widehat{T^k} \frac{1}{\sqrt{2\pi}} = \widehat{T^k} \frac{1}{\sqrt{2\pi}} \chi_E$. Introducing φ as the potential wavelet that arises from the set E, this in turn evaluates to $\widehat{T^k} \widehat{\varphi} = \widehat{T^k \varphi}$. This way, the system $\{\widehat{T^k \varphi} : k \in \mathbb{Z}\}$ is an ONB for the space $L^2(E)$.

Having required E to be also congruent to S modulo dilation by powers of 2, and given that S tiles $\mathbb{R}$ under this group, we see that E is also a dilation-2 tile for $\mathbb{R}$ (this is a simple application of Proposition 1.13). Thus we see that the system $\{L^2(2^j E) : j \in \mathbb{Z}\}$ yields a direct-sum decomposition of $L^2(\mathbb{R})$.

Now, our proof is almost complete: With the functions $\widehat{T^k \varphi}$, $k \in \mathbb{Z}$ being ab ONB for $L^2(E)$, the system $D^j \widehat{T^k \varphi}$ is an ONB for $L^2(2^j E)$ for each $j \in \mathbb{Z}$. Summing over these subspaces, the system $\{D^j \widehat{T^k \varphi} : j, k \in \mathbb{Z}\} = \{\widehat{D^{-j} T^k \varphi} : j, k \in \mathbb{Z}\}$ is a basis for $L^2(R)$. Taking the Fourier transform of the whole system preserves its property of being an orthonormal basis, which leads directly into the definition of φ being a wavelet, as required.

The proof of the converse uses similar methods but is lengthier, drawing in addition on spectral set theory. See [LM06], p.6-8 for a complete argument. □

1.15 Remark. In a recent preprint [ABM07], it was proved that although, in a sense, the translation and dilation groups are "competing", there are actually uncountably many wavelet sets in 1 dimension. For a complete discussion of 1-dimensional wavelet sets see chapter 7.2. of [HW96].

1.2.2 MSF wavelets in n dimensions

Most of the literature on wavelets revolves around the one-dimensional case. Outside of this, of course 2-dimensional wavelets have received the most attention because of applications in image analysis, spawning such fields as curvelets, shearlets, contourlets and other derivatives that take into account the features of common two-dimensional signals (*cf.* [CD02, CDDY06]). In contrast, we shall specifically address the general n-dimensional case.

We make a few modifications to Definitions 1.1 and 1.5 concerning the unitary system $\mathcal{U}$ acting on $L^2(\mathbb{R}^n)$ under which a function generates a basis. First, the system of dilations by powers of 2 is replaced by a system of matrix dilations.

1.16 Definition. A matrix $A \in GL(n, \mathbb{R})$ is called *expansive* if all its eigenvalues are greater than 1 in modulus.

1.17 Remark. Several other definitions equivalent to the above are also in use (*cf.* [LM06], p.3). For example, when verifying that the conditions of Theorem 1.23 apply to our situation, the following definition is useful: A matrix A is called expansive if for every neighbourhood N of 0 and every $r > 0$, there exists an $l \in \mathbb{N}$ such that $B(0, r) \subseteq A^l N$.

We can now replace the dilation operator D, given by $(D\varphi)(t) = \sqrt{2}\varphi(2t)$, by the unitary operator

$$D_A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n); \quad (D_A\varphi)(\mathbf{t}) = |\det A|^{\frac{1}{2}}\varphi(A\mathbf{t}).$$

The translation operator T, given by $(T\varphi)(t) = \varphi(t-1)$, will be replaced by n translation operators T_l in the coordinate directions, given by

$$T_l : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n); \quad (T_l\varphi)(\mathbf{t}) = \varphi(\mathbf{t} - \mathbf{e}_l), \qquad 1 \le l \le n,$$

where $\mathbf{e}_l$ is the lth unit basis vector. Now we are ready to define n-dimensional wavelets.

1.18 Definition. A *dilation-A, regular-translation orthonormal wavelet* is a unit vector $\varphi \in L^2(\mathbb{R}^n)$, such that the system

$$\left\{ D_A^j T_1^{k_1} \dots T_n^{k_n} \varphi : j, k_i \in \mathbb{Z} \right\}$$

is a complete orthonormal basis of $L^2(\mathbb{R}^n)$.

1.19 Remark. Considering lattices other than $\mathbb{Z}^n$ in this definition does not extend the theory: As shown in [ILP98], for a general lattice, there exists a matrix $B \in GL(n, \mathbb{R})$ such that conjugation with B takes the entire wavelet system to the regular-translation system above.

1.20 Definition. A measurable set $E \subseteq \mathbb{R}$ is called a *dilation-A wavelet set* if the scaled characteristic function $\frac{1}{\sqrt{\lambda(E)}}\chi_E$ is the Fourier transform of a dilation-A wavelet.

The characterisation of wavelet sets and associated MSF wavelets in several dimensions, as described by the following theorem, shows little qualitative difference to the preceding characterisation of 1-dimensional wavelet sets. However, showing their actual existence and constructing examples is more involved. Both problems are dealt with here.

1.21 Theorem. *Let $A \in GL(n, \mathbb{R})$ be an expansive matrix. A measurable set $E \subseteq \mathbb{R}^n$ is a dilation-A wavelet set if it tiles $\mathbb{R}^n$ both under the dilation group $\mathcal{D} = \{A^j : j \in \mathbb{Z}\}$ and under the translation lattice $2\pi \cdot \mathbb{Z}^n$.*

Proof. The argument follows the last proof very closely: We consider the set of exponential functions $\{(2\pi)^{-\frac{n}{2}} e^{i\mathbf{k}\cdot\mathbf{s}} : \mathbf{k} \in \mathbb{Z}^n\}$, this time restricted to the n-cube $\prod_{j=1}^n [0, 2\pi)$, on which they form an orthonormal basis. Then these functions are also a basis on our set E, being congruent to this cube via the lattice $2\pi \cdot \mathbb{Z}^n$. For a translation operator T_l, the counterpart $\widehat{T_l}$ in the Fourier domain is given by the modulation $M_{e^{i\mathbf{e}_l\cdot\mathbf{s}}}$. This way, we can again rewrite $\{(2\pi)^{-\frac{n}{2}} e^{i\mathbf{k}\cdot\mathbf{s}} : \mathbf{k} \in \mathbb{Z}^n\}$ as $\{\widehat{T_1^{k_1}} \dots \widehat{T_n^{k_n}} \widehat{\varphi}(\mathbf{s}) : \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n\}$, thereby defining the function φ.
Since E also tiles $\mathbb{R}^n$ under dilations by A-powers, the entire space $L^2(\mathbb{R}^n)$ decomposes into a direct sum of subspaces of functions restricted to $A^j E$. As seen before, this leads to the system $\{\widehat{D_A^{-j}} \widehat{T_1^{k_1}} \dots \widehat{T_n^{k_n}} \widehat{\varphi}\}$ being a basis of $L^2(\mathbb{R}^n)$ (after checking that $\widehat{D_A^j} = D_A^{-j}$), and by taking the inverse Fourier transform, we obtain a wavelet system generated by φ. □

Note that this theorem only gives sufficient conditions for a set to be a dilation-A wavelet set, whereas nothing is said to imply that such a set exists. The non-trivial existence of sets which generate tesselations under matrix dilations and lattice translations is guaranteed by the following theorem by Dai, Larson and Speegle [DLS97]. We consider it worthwhile to quote it in full, after some prefatory definitions.

1.22 Definition. Let X be a metric space, and let m be a σ-finite nonatomic Borel measure on X for which the measure of every open set is positive and for which bounded sets have finite measure. Let $\mathcal{T}$ and $\mathcal{D}$ be countable groups of homeomorphisms of X mapping bounded sets to bounded sets, and mapping m-null sets to m-null sets.

A pair $(\mathcal{T}, \mathcal{D})$ will be called an *abstract dilation-translation pair* if

1. For each bounded set E and each open set F there are elements $\delta \in \mathcal{D}$ and $\tau \in \mathcal{T}$ such that $\tau(E) \subseteq \delta(F)$, and

2. there is a fixed point θ for $\mathcal{D}$ in X which has the property that if N is any neighbourhood of θ and E is any bounded set, there is an element $\delta \in \mathcal{D}$ such that $\delta(E) \subseteq N$.

1.23 Theorem. *Let $X, \mathcal{B}, m, \mathcal{D}, \mathcal{T}$ be as above, with $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair, and with θ the $\mathcal{D}$-fixed point as above. Let E and F be bounded measurable sets in X such that E contains a neighbourhood of θ, and F has nonempty interior and is bounded away from θ. Then there is a measurable set $G \subseteq X$, contained in $\bigcup_{\delta\in\mathcal{D}} \delta(F)$, which is both $\mathcal{D}$-congruent to F and $\mathcal{T}$-congruent to E.*

To use this abstract congruency theorem for our purposes, one can verify that matrix dilation and translation along a lattice actually form an "abstract dilation-translation pair" in terms of the previous definition. For the sets E, F in the theorem, we substitute tiling sets for the group $\mathcal{D}_A$ of dilations by A and the group of translations along $\mathbb{Z}^n$. Such sets are easily found: If B denotes the unit ball in $\mathbb{R}^n$, then the set $F_A := A(B) \setminus B$ is a $\mathcal{D}_A$-tile, and a regular n-cube is a tile for $\mathbb{Z}^n$. We may check that these sets indeed fulfil the conditions imposed on E and F. This way, it is an immediate corollary that there exist sets which are D_A-congruent to F_A and simultaneously $\mathbb{Z}^n$-congruent to the n-cube, and hence constitute dilation-A wavelet sets.

The proof of Theorem 1.23 is constructive, which has prompted several authors to design and publish examples of such two-way tiles. We present two of them in Figures 1.2 and 1.3:

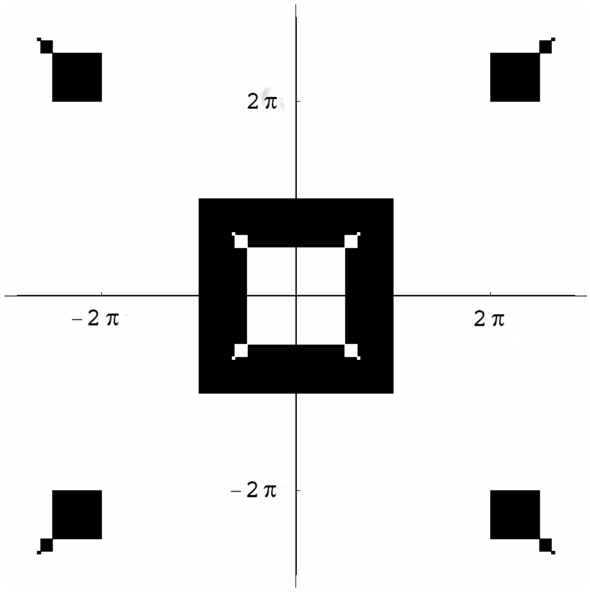


FIGURE 1.2.

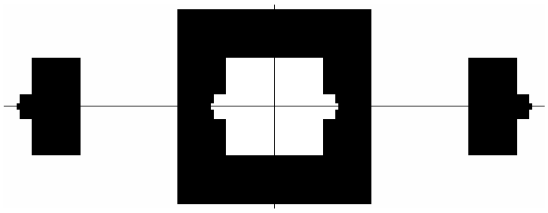


FIGURE 1.3.

The above sets have been called the "four corners set" and the "wedding cake set". Both tile $\mathbb{R}^n$ under translations along $2\pi \cdot \mathbb{Z}^2$ and under the dilation group $\{A^n : n \in \mathbb{Z}\}$ with $A = 2I$. They were presented by X. Dai and D. Larson in [DL98], after some work by P.M. Soardi and D. Weiland had paved the way [SW98], marking an important step forward in finding MSF wavelets in several dimensions.

Chapter 2

Dilation-Reflection Wavelet Sets

In this chapter we consider a different kind of wavelet sets from those treated to far, i.e. dilation-reflection wavelets. As opposed to "traditional" wavelet sets, their construction does not involve Fourier analysis, but the common ground lies in the use of measurable tilings of the underlying space $\mathbb{R}^n$ under a dual dynamical system. This way, the term "wavelet set" as introduced earlier will be assigned a quite different meaning, namely the support of a mother wavelet, rather than the support of its Fourier transform.

In the following sections we will explain, mostly without proofs, how a multiresolution analysis on $\mathbb{R}^n$ can be constructed using fractal functions defined on foldable figures. For details, we refer to [GHMa, GHMb, LM06, Mas95]. The first section treats of the geometric concepts of reflection groups, and the second combines these notions with fractal interpolation functions to yield an MRA structure.

2.1 Reflection Groups and Foldable Figures

We start with a series of definitions, most of them fairly self-explanatory:

2.1 Definition. Given a hyperplane $H \subset \mathbb{R}^n$, determined by a vector r via $H = \{x \in \mathbb{R}^n : \langle x, r \rangle = 0\}$, a (linear) *reflection* about H is a map given by

$$\rho_r(x) = x - \frac{2\langle x, r \rangle}{\langle r, r \rangle}\, r.$$

2.2 Definition. Given an affine hyperplane $H \subset \mathbb{R}^n$, determined by a vector r and a scalar k via $H = \{x \in \mathbb{R}^n : \langle x, r\rangle = k\}$, an *affine reflection* about H is a map given by

$$\rho_{r,k}(x) = x - \frac{2\langle x, r\rangle - k}{\langle r, r\rangle}\, r.$$

Given any finite or infinite set of reflections $\{\rho_i : i \in I\}$, we may use them to generate a finite or infinite *reflection group* $\langle \rho_i : i \in I\rangle$ (of course, a finite set of reflections may generate an infinite group). Merely for completeness, we shall now define so-called Coxeter groups which are an abstract characterisation of reflection groups, while our main concern will be with Weyl groups, constituting a special case of the former.

2.3 Definition. A Coxeter group is an abstract group that admits a presentation of the form

$$\langle r_1, \ldots, r_n | (r_i r_j)^{m_{ij}} = 1\rangle,$$

where $m_{ii} = 1$ for all i, $m_{ij} \geq 2$ for $i \neq j$, and $m_{ij} = \infty$ denotes that no relation exists between ρ_i and ρ_j.

One can see immediately that any reflection group is a Coxeter group. Proceeding towards Weyl groups, the following is an important prerequisite, besides being – despite its apparent special nature – of major importance in Lie theory.

2.4 Definition. A *root system* $\mathcal{R}$ is a finite set of nonzero vectors $r_1, \ldots, r_k \in \mathbb{R}^n$ satisfying

1. $\mathbb{R}^n = \text{span}\{r_1, \ldots, r_k\}$,
2. $r, \alpha r \in \mathcal{R}$ iff $\alpha = \pm 1$,
3. $\forall r \in \mathcal{R} : \ \rho_r(\mathcal{R}) = \mathcal{R}$, i.e., $\mathcal{R}$ is closed under the reflections defined by its elements,
4. $\forall r, s \in \mathcal{R} : \ \rho_r(s) - s \in \mathbb{Z} \cdot r, \quad$ i.e., $\quad \dfrac{2\langle s, r\rangle}{\langle r, r\rangle} \in \mathbb{Z}.$

We note that in some texts (e.g., [Hum90]), the last condition is omitted, and a root system fulfilling it in addition is called a *crystallographic root system* – a term which hints at the rich applications in tesselation problems in three dimensions. It is helpful to immediately list a few examples in two dimensions, namely the systems $A_1 \times A_1$, A_2, B_2, and G_2, which we will meet again in chapter 3:

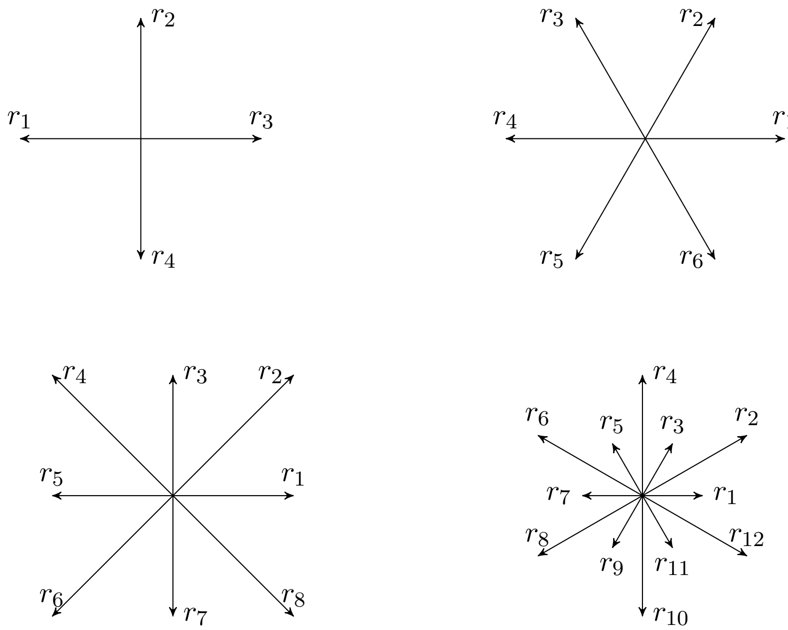


FIGURE 2.1. Root systems $A_1 \times A_1$, A_2, B_2, G_2

We are now ready to define the object which will be ubiquitous in the remainder of this treatise.

2.5 Definition. Given a root system $\mathcal{R}$, the *Weyl group associated to* $\mathcal{R}$ is the reflection group $\mathcal{W}$ given by

$$\mathcal{W} = \langle \rho_r : r \in \mathcal{R} \rangle.$$

From this finite setting, we may take a step of generalisation to include affine reflections:

2.6 Definition. Given a root system $\mathcal{R}$, the *affine Weyl group associated to* $\mathcal{R}$ is the reflection group $\widetilde{\mathcal{W}}$ given by

$$\widetilde{\mathcal{W}} = \langle \rho_{r,k} : r \in \mathcal{R}, k \in \mathbb{Z} \rangle.$$

Clearly the relation between root systems and finite (i.e., non-affine) Weyl groups is one-to-one. A complete classification of root systems has been accomplished, and a common terminology established for them. For an excellent and accessible introduction, see Humphreys' book [Hum90]; another standard text is the volume *Lie Groups and Lie Algebras* of Bourbaki's series *Elements of Mathematics* [Bou02].

Suppose we have two root systems, $\mathcal{R}_1 \subset \mathbb{R}^n$ and $\mathcal{R}_2 \subset \mathbb{R}^m$. Then it is possible to define an obvious "direct product" $\mathcal{R}_1 \times \mathcal{R}_2$ of these, belonging to the product space $\mathbb{R}^{m+n}$, such that each root $r_i \in \mathcal{R}_1$ is perpendicular to every root $r_j \in \mathcal{R}_2$. This way, $\mathcal{R}_1 \times \mathcal{R}_2$ indeed satisfies the defining properties of a root system. The system $A_1 \times A_1$ in Fig. 2.1 is an example of such a product. A root system (and its associated Weyl group) which cannot be represented in this way as a direct product of lower-dimensional systems will be called *essential*.

When classifying root systems, we may thus restrict ouselves to essential systems. These have been grouped into a relatively small number of classes, denoted by capital letters A to G, together with a number denoting the dimension. We have the classes

- A_n with $n \geq 1$,
- B_n with $n \geq 2$,
- C_n with $n \geq 3$,
- D_n with $n \geq 4$,
- $E_6, E_7, E_8; F_4; G_2$.

There also exist representations of root systems by means of so-called Dynkin diagrams, which are directed multigraphs whose connected components represent the "essential factors" of a root system, while in a similar way, the more general Coxeter groups may be represented by means of so-called Coxeter graphs; however this is only tangentially relevant to our aims here.

Lastly, we need to define a geometric object that will form the "basic cell of self-similarity" of the fractal functions we will introduce in the next section.

2.7 Definition. A compact connected subset F of $\mathbb{R}^n$ is called a *foldable figure* if there exists a finite set $\mathcal{S}$ of affine hyperplanes that cuts F into finitely many congruent subfigures $F_1, \ldots, F_m$, each similar to F, so that reflection in any of the cutting hyperplanes in $\mathcal{S}$ bounding F_k takes it into some F_l.

Foldable figures and Weyl groups are closely related, as shown by the following theorem:

2.8 Theorem. *There exists a bijection between the set of all essential affine Weyl groups (in all dimensions) and the set of all foldable figures. Each foldable figure is the fundamental domain of the Weyl group that corresponds to it via this bijection.*

2.2 Construction of Fractal Interpolation Functions

The use of fractal funtions in interpolation problems was first developed by Barnsley in 1986 [Bar86]. Subsequent publications on this topic include the paper by Massopust [Mas90], the work by Geronimo/Hardin [GH93], and the recent paper by Larson/Massopust [LM06]. We aim to construct fractal functions on simplices in $\mathbb{R}^n$ which interpolate given values at their vertices. Like most fractal objects, the fractal funtions we consider are obtained via a limiting process.

In [LM06], fractal functions are introduced via so-called *iterated function systems*. Using this method, a fractal is defined as a limit of compact sets. Thus we need a notion of convergence of sets, which is provided by the following metric:

2.9 Definition (Hausdorff distance)**.** Let X and Y be non-empty compact subsets of a metric space (M,d). Then the Hausdorff distance between X and Y, denoted d_H, is defined as

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

This gives rise to the metric space of non-empty compact subsets of M, which will be denoted by $\mathcal{K}(M)$.

Hausdorff metric has a number of useful properties, which we will need again in chapter 3:

1. If the underlying metric space M is complete, then so is $\mathcal{K}(M)$. Similarly, if M is totally bounded, then so is $\mathcal{K}(M)$.

2. Lebesgue measure is continuous with respect to limits in Hausdorff metric. In precise terms, if $(A_n)_{n\in\mathbb{N}} \subseteq M$ is a Hausdorff-convergent sequence of compact sets, then $\lambda(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \lambda(A_n)$.

2.10 Definition. Let (X,d) be a complete metric space with metric d, and let $\{T_i : i = 1,\dots,N\}$ be a finite set of contractions on X. The pair $((X,d),\{T_i\})$ is called an *iterated function system (IFS)* on X.

Consider an IFS $((X,d),\{T_i\})$ on a metric space X, and define the space $\mathcal{K}(X)$ according to Definition 2.9. We may then define a contractive operator $\mathcal{J} : \mathcal{K}(X) \to$

$\mathcal{K}(X)$, called the *Hutchinson operator*, given by

$$\mathcal{J}(E) = \bigcup_{i=1}^{N} T_i(E).$$

One can show that the Hutchinson operator is contractive on $\mathcal{K}(X)$ with contractivity constant $\max_{1\leq i\leq N} s_i$, where s_i is the contractivity constant of T_i. By the above listed properties of Hausdorff metric, $\mathcal{K}(X)$ is a complete metric space, so we may apply the Banach fixed point theorem to show that T has a unique fixed point, called the *fractal* $\mathfrak{F}$ *associated with the IFS* $((X, d), \{T_i\})$. The proof of the Banach fixed point theorem also shows that $\mathfrak{F}$ can be obtained as the limit of the sequence $F_n = \mathcal{J}^n(F_0)$, i.e. the n-fold application of $\mathcal{J}$ to any set $F_0 \in \mathcal{K}(X)$.

Given this last point, it is not hard to define iterated functions systems on $\mathbb{R}^n$ and produce demonstrative depictions of continuous fractal functions through explicit computation of a finite term of the above sequence. A related but more generalised method is to obtain fractal functions as fixed points of so-called Read-Bajraktarević operators. We will need the following result about a particular kind of such operators:

2.11 Theorem. *Let* $\Omega \subset \mathbb{R}$ *be compact and* $1 < N \in \mathbb{N}$. *Assume that* $u_i : \Omega \to \Omega$ *are contractive homeomorphisms inducing a partition on* Ω, $\lambda_i : \mathbb{R} \to \mathbb{R}$ *are bounded functions and* s_i *real numbers,* $i = 1, \ldots, N$. *Let*

$$\mathscr{B}(f) := \sum_{i=1}^{N} [\lambda_i + s_i f] \circ u_i^{-1} \chi_{u_i(\Omega)} \,.$$

If $\max\{|s_i|\} < 1$, *then the operator* $\mathscr{B}$ *is contractive on* $L^\infty(\Omega)$ *and has a unique fixed point* $\mathfrak{F} : \Omega \to \mathbb{R}$.

Proof. Easily shown using the Banach fixed point theorem. □

The fixed point obtained this way is an alternative way of defining a fractal function. An advantage of using operators like the above is that when defined on function spaces with additional regularity properties, under certain conditions the fixed points will be elements of the same space. This way, fractal functions with prescribed regularity may be obtained [Mas05].

We will now explain the construction of a fractal surface on a particular foldable figure in two dimensions; it will be clear how this can be transferred to Weyl groups and associated foldable figures of arbitrary dimension.

Let $\widetilde{W}$ be the affine Weyl group associated to the root system B_2. The fundamental domain of $\widetilde{W}$ is an isosceles triangle with vertices $(0,0),(1,0),(0,1)$, which shall be denoted by Δ. Being a foldable figure, it may be partitioned into four congruent triangles $\Delta_i, i=1,\ldots,4$, each similar to Δ. The similitudes may be expressed by four affine-linear mappings $u_i : \Delta \to \Delta_i$ of the form $\sigma O_i + b_i$, the O_i being orthogonal matrices, σ being the similarity ratio, and $b_i \in \mathbb{R}^n$.

We now define four affine-linear functions λ_i on Δ (i.e., first degree polynomials), such that the composition $\sum_{i=1}^4 \lambda_i \circ u_i^{-1}$ is a continuous, piecewise affine-linear function on Δ. Note that this function is completely determined by its values on the six vertices of the Δ_i (i.e. the "outer" vertices $(0,0)^\mathsf{T},(1,0)^\mathsf{T},(0,1)^\mathsf{T}$ and the "inner" vertices $(\frac{1}{2},0)^\mathsf{T},(0,\frac{1}{2})^\mathsf{T},(\frac{1}{2},\frac{1}{2})^\mathsf{T}$) – in other words, we are considering a piecewise-linear interpolation function on six support points. Evidently the space of such interpolation functions is then six-dimensional.

Now fix a real number s with $|s|<1$, to serve as contractivity constant of a contractive mapping, and define the operator

$$\mathscr{B}f(x) = \sum_{i=1}^{4} [\lambda_i + sf] \circ u_i^{-1}(x)\chi_{\Delta_i}(x).$$

By the previous theorem, $\mathscr{B}$ has a unique fixed point, the fractal $\mathfrak{F} = \mathscr{B}(\mathfrak{F})$. Fig. 2.2 shows the graph of such a function on our chosen Δ for a particular choice of functions λ_i.

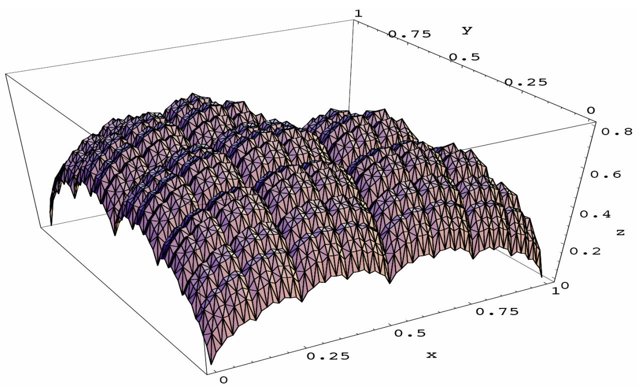


FIGURE 2.2.

Note that on the six outer and inner vertices, $\mathfrak{F}$ takes the same values as the function $\sum_{i=1}^4 \lambda_i \circ u_i^{-1}$, i.e. it is an interpolation function, too. Moreover, since $\mathfrak{F}$ only depends on the functions $\{\lambda_i : i = 1, \ldots, 4\}$ (suppressing the dependency on the constant s for the moment) which are in turn determined by the interpolation points, there exists a linear isomorphism from the set of tuples $(\lambda_1, \ldots, \lambda_4)$ to the space of fractal functions on Δ, and this space is also six-dimensional. Therefore, if $v_1, \ldots, v_6$ denote the six vertices, there exists a basis of fractal functions $\{\varphi_i\}$ for this space satisfying

$$\varphi_i(v_j) = \delta_{ij}.$$

As shown in [Mas95], this basis may also be orthonormalised using the Gram-Schmidt procedure.

We note that so far, no loss of generality was entailed in our choice of Δ. From now on we consider a general foldable figure Δ, partitioned into N subfigures.

We now make two extensions to this setting: Firstly, the "generating functions" $\lambda_i, i = 1, \ldots, N$ may be taken to be polynomials of degree n rather than 1; in notation, let us write $\lambda_i \in \Pi^d(\Delta)$. Remembering the restriction that the functions λ_i have to satisfy a join-up condition to give rise to a continuous fractal, we denote the space of such functions by J^d. This way, we may say that a general fractal function defined on Δ is determined by a vector of functions $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (J^d)^N$. With this, Theorem 2.11 may still be applied and a wider class of fractals obtained.

More importantly, by the tesselation property of Δ, we can define fractal function spaces with associated orthonormal bases on each copy $r\Delta$ of Δ under $\widetilde{\mathcal{W}}$, hence obtaining a space of *global fractal functions*. Thus, while an element of the space of fractal functions defined on Δ was determined by a vector $\boldsymbol{\lambda} \in J^d$, a global fractal function is uniquely specified by one such vector for each $\widetilde{\mathcal{W}}$-copy of Δ. We denote by $\boldsymbol{\Lambda}$ such a collection of vectors of functions, and by $J^{\widetilde{\mathcal{W}}}$ the space of such collections. Thus we have an obvious bijection between $J^{\widetilde{\mathcal{W}}}$ and the set of global fractal functions generated by the action of $J^{\widetilde{\mathcal{W}}}$, associating to each $\boldsymbol{\Lambda} \in J^{\widetilde{\mathcal{W}}}$ a fractal $\mathfrak{F}_{\boldsymbol{\Lambda}}$.

2.3 Dilation-Reflection Multiresolution Analysis

We are now ready to modify Definition 1.4 for dilation and reflection groups. In the following, let $q > 1$ be a positive integer, and D_q the dilation operator given by the matrix $q \cdot I$.

2.12 Definition. A multiresolution analysis for a dilation-reflection pair $D_q, \widetilde{\mathcal{W}}$ is a sequence of closed subspaces of $L^2(\mathbb{R}^n)$ satisfying the following properties:

1. $V_j \subset V_{j+1}$ for all $k \in \mathbb{Z}$;
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
3. there exists a finite set of generators $\{\varphi | a \in A\}$ such that
$$\mathcal{B}_\varphi := \{\varphi^a \circ r | a \in A, r \in \widetilde{\mathcal{W}}\}$$
is an orthonormal basis for V_0;
4. $f \in V_j \Leftrightarrow D_q f \in V_{j+1}$.

As yet we haven't indicated whether such an MRA may exist. In fact, it can be shown that by letting

$$\begin{aligned} V_0 &= \{\mathfrak{F}_{\mathbf{\Lambda}} : \mathbf{\Lambda} \in J^{\widetilde{\mathcal{W}}}\}, \qquad \text{and} \\ V_k &= D_q^k V_0, \end{aligned}$$

the above conditions are verified. For details on why these properties hold, we refer to [LM06, GHMb, Mas95].

To obtain a wavelet from the MRA now established, the approach will be similar to the one followed for ordinary wavelets, as described on p. 12: An orthonormal basis $\{\psi^l : 1 \leq l \leq s\}$ may be found for the *wavelet space* W_0 restricted to Δ (as before, W_0 being the complement space $V_1 \ominus V_0$), so that the system $\mathcal{B}_{\boldsymbol{\psi}} := \{\psi^l \circ r : 1 \leq l \leq s,\ r \in \widetilde{\mathcal{W}}\}$ yields an ONB for the whole of W_0. By the decomposition of $L^2(\mathbb{R}^n)$ into orthogonal subspaces W_j, the D_q-dilates of this basis then give a complete ONB of $L^2(\mathbb{R}^n)$, as desired.

2.4 Existence of Dilation-Reflection Wavelet Sets

The required orthogonality of the system $\{D_q^k \mathcal{B}_{\boldsymbol{\psi}} | k \in \mathbb{Z}\}$ implies that its elements are supported on disjoint subsets of $\mathbb{R}^n$. This brings back the topic of tesselations. More precisely, it is necessary that the union of the supports of the generators $\{\psi^l \circ r : 1 \leq l \leq s,\ r \in \widetilde{\mathcal{W}}\}$ be a two-way tile for the pair $(\mathcal{D}, \widetilde{\mathcal{W}})$. Such a two-way tile is what we mean by a *dilation-reflection wavelet set.* This is a conspicuous similarity with the characterisation of n-dimensional wavelet sets in Theorem 1.21.

It comes as a pleasant surprise that this similarity also extends to the solution of this tesselation problem: Probably unforeseen by its authors, Theorem 1.23 also shows the existence of dilation-reflection wavelet sets. As a reminder, this theorem essentially shows that two-way tiles exist for certain "abstract dilation-translation pairs". As it happens, any dilation-reflection pair $(\mathcal{D}, \widetilde{\mathcal{W}})$ fulfils the conditions for being such a pair. We put this as a proposition:

2.13 Proposition. Let $\mathcal{D}$ be a group of matrix dilations, and $\widetilde{\mathcal{W}}$ an affine Weyl group. Let C be the fundamental domain of $\widetilde{\mathcal{W}}$ (i.e. a foldable figure), and choose $F_A := A(B(0,1)) \setminus B(0,1)$ as a fundamental domain for $\mathcal{D}$ (as before). Then $(\mathcal{D}, \widetilde{\mathcal{W}})$ is an abstract dilation-translation pair, as defined prior to Theorem 1.23. In addition, C and F_A fulfil the conditions of the sets E, F as put forth in the statement of this theorem.

Corollary. There exist $(\mathcal{D}, \widetilde{\mathcal{W}})$-tiles (i.e., dilation-reflection wavelet sets) for every pair $(\mathcal{D}, \widetilde{\mathcal{W}})$.

These results pave the way for the construction of dilation-reflection wavelet sets similar to the way "traditional" dilation-translation wavelet sets have been obtained.

In fact there is yet another accomplishment in the field of traditional wavelet sets that can be recycled: It was noticed that the "four-corners set", depicted in Fig. 1.2, which had been constructed as a tile for the dilation-translation pair $(2I, \mathbb{Z}^n)$, is also a tile for the dilation-reflection pair $(2I, A_1 \times A_1)$. Hence, this set constitutes both a regular n-dimensional wavelet set and a dilation-reflection wavelet set.

All these facts taken together suggest that it might be rewarding to further investigate the links between these two wavelet theories. Our results in the next chapter also belong to this intersection.

Chapter 3

Triple wavelet sets

In this chapter we will motivate, define and give examples of triple wavelet (or three-way tiling) sets in $\mathbb{R}^n$. Section 3.2 states and proves the main result, i.e. the existence of triple wavelet sets for a wide class of triples $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$ of affine Weyl groups, expanding matrices, and full-rank lattices. In particular, triple wavelet sets exist for any affine Weyl group. Section 3.3 lists a few examples of three-way tiling sets in two dimensions. In the explicit construction of such sets, we noticed a link to the third of Hilbert's famous set of 23 problems; this relation is explored in section 3.4.

3.1 Motivation

At the end of the first chapter, we have given an example of a dilation-translation wavelet set in two dimensions. In chapter 2, we saw that the same set is also a dilation-reflection wavelet set. Thus, this set is the first example of what we call a *triple wavelet set* for the triple $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$. It is natural to ask whether there are other examples of such sets.

3.2 Main Result: Existence of triple wavelet sets in any dimension

In this section we prove the following theorem, giving a complete positive answer to subproblem 2b als formulated in [LMÓ07], p. 17.

3.1 Theorem. *Let $\widetilde{\mathcal{W}}$, $\widetilde{\Gamma}$ be an affine Weyl group and a translation group with equal measure of fundamental domains, such that the intersection $\widetilde{\mathcal{W}} \cap \widetilde{\Gamma}$ contains a full-rank lattice. Further, let A be an expanding matrix. Then there exists a $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$-triple wavelet set, i.e. a measurable set $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ which tiles $\mathbb{R}^n$ under the operation of all three groups.*

Following our method, the proof is inevitably lengthy and technical. The author believes that for clarity, it is beneficial to retreat to a slightly weaker result, and to defer the modifications for a full generalisation of the proof to a concluding remark. Thus we shall prove the following:

3.1' Theorem. *Let $\widetilde{\mathcal{W}}, A, \widetilde{\Gamma}$ be as above, requiring in addition that $\widetilde{\Gamma} = D\Gamma$, where Γ is the coroot lattice of $\widetilde{\mathcal{W}}$ and D is a diagonal matrix with rational entries. That is, the vectors spanning $\widetilde{\Gamma}$ are rational multiples of those spanning Γ. Then there exists a $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$-triple wavelet set.*

It is clear that the latter result is a special case of the former. The choice of $\widetilde{\Gamma}$ may be special, but this case is still by far sufficient to show the existence of triple wavelet sets in any dimension, and for any Weyl group.

The proof proceeds as follows. We begin with some terminology to simplify the argument. We then reduce the problem to that of finding two-way tiles for pairs $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ (section 3.2.2). Next, we construct such two-way tiles for certain special pairs that do not share a full-rank lattice (section 3.2.3). Then, by means of two limiting operations over such pairs, we obtain two-way tiles for systems $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ that do have this "commensurability property" (section 3.2.4). Due to the limits taken in the process, the proof does not lend itself to explicit constructions.

3.2.1 Preliminaries

Commensurability. This gives a short-hand name to the property of two transformation groups sharing a full-rank sublattice:

3.2 Definition. Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^n$ be two full-rank lattices. If $\Gamma_1 \cap \Gamma_2$ contains another full-rank lattice, we will say that Γ_1, Γ_2 are *commensurable*; otherwise the lattices are called *incommensurable*. If the intersection is empty, the lattices are called *totally incommensurable*. We will also say that the pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ is commensurable if $\widetilde{\Gamma}$ is commensurable with Γ, the coroot lattice of $\widetilde{\mathcal{W}}$.

3.3 Remark. We will use the term "lattice" ambiguously to describe either a translation group, or a point set in $\mathbb{R}^n$ – usually, no confusion should be possible.

Reduction modulo a group of transformations. Given a group of measurable transformations, we define a function that maps any measurable set onto a corresponding subset of the fundamental domain of the group.

3.4 Definition. Let G be a group of measurable transformations of $\mathbb{R}^n$, and let K be a G-tile. Then we can define a function $r_G : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(K)$, called *reduction modulo* G, which maps every Borel set X uniquely (up to Lebesgue null sets) to a subset $r_G(X)$ of K that has a partition $r_G(X) = \bigcup_{g \in G} X_g$ such that $X = \bigcup_{g \in G} g(X_g)$. That is, every point in X is mapped into K by some suitable element of G.

G-compatibility and partial tiles. We need a name for measurable sets which are smaller in measure than the fundamental domain of a transformation group, but which can possibly be extended to a tile for the group (or, as below, may be used as elements of a sequence of sets of increasing measure that converge to a tile).

3.5 Definition. As above, let G be a group of transformations and K a G-tile. A measurable set S will be called *G-compatible*, or a *G-partial tile*, if $\lambda(S) = \lambda(r_G(S))$.

More graphically, if S is G-compatible, no "overlaps" of positive measure occur when reducing S modulo G. In yet other words, the family of sets $\{gS : g \in G\}$ is disjoint modulo null sets.

3.2.2 Reduction of problem to two groups of transformations

As a first step, we present a result by which our task is reduced to finding reflection-translation sets. This is an adaptation of the result 8.1 at the end of [LM06], which in turn draws on Theorem 1.23 (Theorem 1 of [DLS97]).

3.6 Lemma. Consider a triple $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$ such that the pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ is commensurable. Suppose there exists a measurable two-way tile $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$ for the pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ which contains 0 in its interior. Then there exists a three-way tile $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ for the triple $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$.

Proof. Let Π denote the "intersection lattice" contained in $\widetilde{\mathcal{W}} \cap \widetilde{\Gamma}$. Let B denote the unit ball in $\mathbb{R}^n$, and let $F = A(B) \setminus B$. Then the sets $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}, F$ fulfil the conditions on the sets E, F in Theorem 1.23, and the groups $\mathcal{D} = \{A^n : n \in \mathbb{Z}\}, \mathcal{T} = \Pi$ constitute an abstract dilation-translation pair as in this theorem. Thus the theorem concludes that there exists a measurable set $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ which is (i) $\mathcal{D}$-congruent to F, and (ii) Π-congruent to $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$.

Now (i) implies that $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ is a dilation A-tile (since F is such a tile, too). Concerning (ii), since Π is a subgroup of both $\widetilde{\mathcal{W}}$ and $\widetilde{\Gamma}$, the set $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ is, in particular, both $\widetilde{\mathcal{W}}$- and $\widetilde{\Gamma}$-congruent to $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$. This means that $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ tiles $\mathbb{R}^n$ under both $\widetilde{\mathcal{W}}$ and $\widetilde{\Gamma}$. These three tiling properties make $W_{\widetilde{\mathcal{W}},A,\widetilde{\Gamma}}$ a triple wavelet set for $(\widetilde{\mathcal{W}}, A, \widetilde{\Gamma})$. □

Now, it suffices to find two-way tiles for commensurable pairs $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ of reflection and translation groups. We point out that the condition that such a two-way tile should contain 0 in its interior may be omitted, which we will do in the following for the sake of clarity, although in fact the remainder of the proof could easily be modified to fulfil it. The reasons that it may be omitted are technical and contribute little to this disussion; for details we refer to [LMÓ07], pp.7-10.

3.2.3 Two-way tiles for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$, incommensurable case

Surprisingly, it has proved to be easier to find two-way tiles for the case where the pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ (or rather $(\Gamma, \widetilde{\Gamma})$, where Γ is the coroot lattice) is incommensurable. We have the following result:

3.7 Proposition. Let $\widetilde{\mathcal{W}} = \mathcal{W} \ltimes \Gamma$ be an affine Weyl group acting on $\mathbb{R}^d$. Then there exists a full-rank lattice $\widetilde{\Gamma}$ that is totally incommensurable with Γ, and a measurable set S which tiles $\mathbb{R}^d$ under both $\widetilde{\mathcal{W}}$ and $\widetilde{\Gamma}$.

Before the proof, we will describe intuitively the advantage of incommensurability. First, consider a simple one-dimensional situation: Let r be an irrational real number between 0 and 1. Now, consider the result of multiplying r by the set of integers $\mathbb{Z}$, and taking fractional parts of the resulting set of irrational numbers. That is, we examine the set $(\mathbb{Z} \cdot r) \bmod 1$. It is elementary analysis that this set is dense in the interval $[0, 1]$.

Now, we can generalise this idea to several dimensions. What we have done above is to consider two incommensurable lattices, $\Gamma_1 = \mathbb{Z}$ and $\Gamma_2 = \mathbb{Z} \cdot r$. The fact that $(\mathbb{Z} \cdot r) \bmod 1$ is dense in $[0, 1]$ is just another way of saying that $r_{\Gamma_1}(\Gamma_2)$

is dense in the fundamental domain of Γ_1. Instead we can now take any full-rank lattice $\Gamma_1 = M_1\mathbb{Z}^n$, where $M_1 \in GL_n(\mathbb{R})$, and another lattice $\Gamma_2 = M_2\mathbb{Z}^n$ such that $M_2 = DM_1$, where D is a diagonal matrix with irrational entries. So each "basis vector" of Γ_2 is an irrational multiple of a corresponding basis vector of Γ_1. Then, in complete analogy to the one-dimensional case, $r_{\Gamma_1}(\Gamma_2)$ is dense in the fundamental domain of Γ_1. The significance of this in constructing two-way tiles is that by taking suitable elements of Γ_2, we can get arbitrarily close to any point in the fundamental domain of Γ_1.
The main part of the proof follows.

Proof of Proposition 3.7. Deviant from the notation employed in [LMÓ07], let R denote the fundamental cell of $\widetilde{\mathcal{W}}$, T the fundamental cell of $\widetilde{\Gamma}$, whereas (in accordance with [LMÓ07]), K shall stand for the fundamental cell of Γ.

As described above, let us choose the lattice $\widetilde{\Gamma}$ as the $\mathbb{Z}$-span of irrational multiples of the coroots spanning Γ – or, in oversimplified notation, $\widetilde{\Gamma} = D\Gamma$, where D is a diagonal matrix with irrational entries. There remains one more restriction on $\widetilde{\Gamma}$ – we have to ensure that $\lambda(T) = \lambda(R)$. Given that $\lambda(R) = \lambda(K)/|\widetilde{\mathcal{W}}|$, it suffices to choose D such that $\det D = |\widetilde{\mathcal{W}}|^{-1}$.

We will construct a sequence $(S_n)_{n=1}^{\infty}$ of closed sets whose union $S = \bigcup_{n\geq 1} S_n$ is a two-way tiling set for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$. That is, S must be $\widetilde{\mathcal{W}}$-congruent to R and $\widetilde{\Gamma}$-congruent to T. We set $S_1 = \emptyset$, and proceed by adding in each step a closed ball such that the next S_n thus obtained remains $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-compatible – that is, no overlap occurs within the family of sets $\widetilde{\mathcal{W}}S_n$, nor within the family $\widetilde{\Gamma}S_n$. Let us write R_n for $r_{\widetilde{\mathcal{W}}}(S_n)$, and T_n for $r_{\widetilde{\Gamma}}(S_n)$.

At this point, note that T can be written as a union of closed balls centered at points in T with rational $\widetilde{\Gamma}$-coordinates. These points are countable, so we can enumerate them as a sequence $(a_n)_{n=1}^{\infty}$. Having constructed S_n, if a_n already lies in T_n, we set $S_{n+1} = S_n$. Otherwise, we add to S_n some closed ball centered at a_n or one of its $\widetilde{\Gamma}$-translates which does not intersect $\widetilde{\mathcal{W}}R_n$. Proceeding this way to construct the family $(S_n)_{n=1}^{\infty}$ (and hence the families $(R_n)_{n=1}^{\infty}, (T_n)_{n=1}^{\infty}$), the family (T_n) will exhaust T, and since $\lambda(T) = \lambda(R)$ and $\lambda(T_n) = \lambda(R_n)$, the R_n will also exhaust R. It just remains to show that in each step, a suitable ball (i.e. a suitable translate with suitable radius) can always be found to be added to S_n.

Suppose the contrary, i.e. that in the nth step, all the translated balls $B(a_n, r)+$

$\widetilde{\Gamma}$, for any radius r, intersect $\widetilde{\mathcal{W}} R_n$. This means that the points of the *affine lattice* $a_n + \widetilde{\Gamma}$ are accumulation points of $\widetilde{\mathcal{W}} R_n$. But in the neighbourhood of any such lattice point, we need only to consider a finite number of copies of R_n, whose union is closed (as R_n is closed, a property fulfilled for R_1 and preserved in each step of the construction). As accumulation points of closed sets, the elements of $a_n + \widetilde{\Gamma}$ are in fact contained in them, so we have

$$a_n + \widetilde{\Gamma} \subseteq \widetilde{\mathcal{W}} R_n = \mathcal{W} R_n + \Gamma \,,$$

and hence

$$r_\Gamma(a_n + \widetilde{\Gamma}) \subseteq \mathcal{W} R_n \subseteq K \,.$$

But since $\widetilde{\Gamma}$ has been defined so that Γ and $\widetilde{\Gamma}$ are totally incommensurable, $r_\Gamma(\widetilde{\Gamma})$ is dense in K, and so must be $r_\Gamma(a_n + \widetilde{\Gamma})$. This is the crucial point in the argument. Thus $\mathcal{W} R_n$ is dense in K, i.e. R_n is dense in R. This contradicts the fact that $\forall n$, the "remainder yet uncovered by our set", $R \setminus R_n$, always contains an open ball (this holds for $R_1 = \emptyset$ and is preserved in passing from R_n to R_{n+1}). The proof is complete. □

3.8 Remark. Note that the boundary of any open or closed subset of $\mathbb{R}^n$ has n-dimensional Lebesgue measure zero. Thus, taking the closure or interior of a "partial tile" neither changes its measure, nor its $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-compatibility. Thus, at any point in our construction, we can assume w.l.o.g. that our (partial) tiling sets S_n are either closed or open. This fact will also be useful in the following section.

3.2.4 Two-way tiles for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$, commensurable case

We are now ready to proceed to show the existence of $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-tiles in the commensurable case. With Lemma 3.6, the desired existence of three-way tiling sets is then immediate. As a formal statement, we have

3.9 Proposition. Let $\widetilde{\mathcal{W}} = \mathcal{W} \ltimes \Gamma$ be an affine Weyl group, and $\widetilde{\Gamma}$ a lattice commensurable with Γ. Then there exists a two-way tile for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$.

The initial idea behind the proof is the following: Given $\widetilde{\mathcal{W}}$, we construct a sequence of Γ-incommensurable lattices $(\widetilde{\Gamma}_n)_{n=1}^\infty$ which in some sense converge to a Γ-commensurable lattice $\widetilde{\Gamma}$. By Proposition 3.7, we can find a two-way tile for each pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma}_n)$. But these tiles are potentially unbounded, and it is hard to imagine a sequence of unbounded sets converging to a two-way tile for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$, which, if it exists, can always be assumed to lie within the compact fundamental domain of the lattice shared by Γ and $\widetilde{\Gamma}$. Therefore, more effort is necessary: We carry out two consecutive limiting processes, explained in the following.

First limiting process

Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of irrational numbers converging to 1 from below, and let $\widetilde{\Gamma}_n = \beta_n \widetilde{\Gamma}$. By Theorem 3.7, we can find a tile for each pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma}_n)$, which we shall denote by U_n. Note however that, since the fundamental domains of the $\widetilde{\Gamma}_n$ have smaller measure than the reflection tiling set, R, the U_n that we can construct are only "partial tiling sets": They are both $\widetilde{\Gamma}_n$- and $\widetilde{\mathcal{W}}$-compatible, but $r_{\widetilde{\mathcal{W}}}(U_n)$ is a subset of R that does not have full measure.

As yet, there is no way that the U_n can converge in any sense, or merely possess a convergent subsequence. To make use of the Hausdorff metric, the U_n would have to be contained in a compact subset of $\mathbb{R}^n$, which cannot be assumed. Therefore, we restrict our family of partial tiles to a large closed ball $K(r)$ of radius r: Let

$$U_n^r = U_n \cap K(r).$$

Note that, by Remark 1 of section 3.2.3, the U_n may be assumed to be closed. Therefore, the U_n^r are closed too, and hence compact.

Let $\mathcal{K}(r)$ denote the space of compact subsets of $K(r)$. Since $K(r)$ is complete and totally bounded, so is $\mathcal{K}(r)$ (as explained after Definition 2.9, property (1)). Thus, $\mathcal{K}(r)$ is also sequentially compact, which is essential: Now the sequence $(U_n^r)_{n=1}^{\infty}$ has a convergent subsequence, with limit U^r, say.

Now the aim would be to find a limit of the sets U^r for $r \to \infty$. But so far, we cannot control the measure of each U^r: We cannot guarantee yet that $\lambda(U^r)$ increases to $\lambda(R)$, the measure of the fundamental cell of the Weyl group. So we make a modification to the construction in the proof of Theorem 3.7, making sure that the mass of the two-way tiles is concentrated near the origin.

As a reminder, this construction involved a nested sequence of "partial sets" S_n, where S_{n+1} was obtained from S_n by adding a small closed ball. The union $S = \bigcup_{n \geq 1} S_n$ then constitutes the two-way tile. Now, let r^* be the radius of a ball large enough to contain the fundamental cell of Γ, the coroot lattice. For any $d > r^*$, we will try to make sure that no more than a certain proportion of the mass of S will end up outside a ball of this radius - say, no more than $\lambda(S)/d$. Thus, as we pass from S_n to S_{n+1}, if a ball to be added to the partial tile is centered at a point of a distance $d > r^*$ from the origin, we assess how much mass of the set S_n is already located further out, say $\lambda_{\geq d}(S_n)$. Then we choose the radius of the ball to be added such that its measure m_b satisfies

$$m_b < \frac{\lambda(S)/d - \lambda_{\geq d}(S_n)}{2}.$$

This way, for all n and $d > r^*$, the "outer mass" $\lambda_{\geq d}(S_n)$ will never exceed $\lambda(S)/d$, and the mass distribution of S will decay with increasing distance from the origin in a prescribed way.

Returning to our compact partial two-way tiles U_n^r with this modification made, we can be sure that these sets possess a certain minimal measure; that is,

$$\lambda(U_n^r) \geq (1 - r^{-1})\ \lambda(R) \qquad \text{for } r > r^*,\ \forall n.$$

Since Lebesgue measure is continuous with respect to Hausdorff limits (as noted after Definition 2.9), we can also be sure that the limit set U^r satisfies $\lambda(U^r) \geq (1 - r^{-1})\ \lambda(R)$. There just remains an easy step to show that U^r is also $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-compatible.

Indeed, assuming that U^r is not $\widetilde{\Gamma}$-compatible quickly leads to a contradiction. Suppose that a positive measure subset of U^r can be mapped into U^r by a nonzero element t of $\widetilde{\Gamma}$. Pick an open ball $B = B(a, \varrho)$ inside this set – so both B and $B + t$ belong to U^r. Now, fix an index n satisfying two conditions: Firstly, $d_H(U_n^r, U^r) < \frac{\varrho}{4}$, and further, $(1 - \beta_n)\|t\| < \frac{\varrho}{4}$. The first condition ensures that the annulus $B \setminus B(a, \frac{\varrho}{4})$ and its translate both belong to U_n^r. The second one guarantees that a ball B' of radius $< \frac{\varrho}{4}$ can be found inside $B \setminus B(a, \frac{\varrho}{4})$ such that $B' + \beta_n t \subset \big(B \setminus B(a, \frac{\varrho}{4})\big) + t \subset U_n^r$. So both B' and $B' + \beta_n t$ belong to U_n^r and have positive measure, which is a contradiction, since U_n^r is $\widetilde{\Gamma}_n$-compatible. This geometric argument is illustrated in the following figure – the region $B \setminus B(a, \frac{\varrho}{4})$ and its t-translate are lightly shaded, the darker circles are B' and $B' + \beta_n t$.

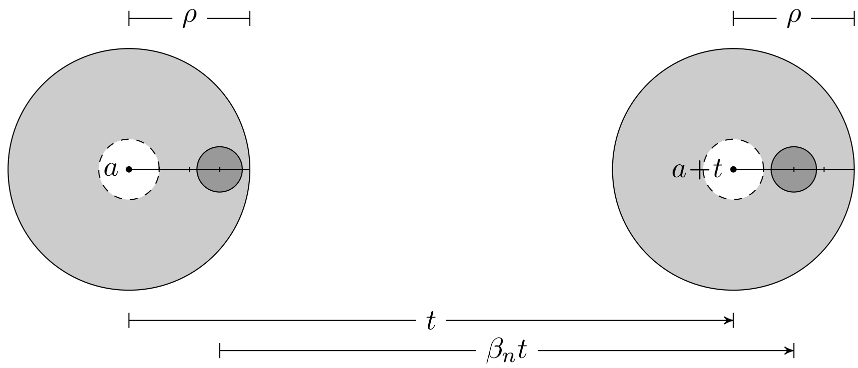


FIGURE 3.1.

Thus it is proved that the limit set U^r is $\widetilde{\Gamma}$-compatible – very similar geometric considerations show $\widetilde{\mathcal{W}}$-compatibility. With this, the goal of this subsection is reached.

Second limiting process

So far, we have been able to engineer a family $\{U^r : r \in \mathbb{R}, r > r^*\}$ of sets with the following properties:

1. They are compact sets, as subsets of the Hausdorff-metric space $\mathcal{K}(r)$.
2. They are "partial" (i.e., not full measure) tiles for the pair $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$.
3. As r increases, their measure approaches $\lambda(R) = \lambda(T)$ (the measure of the fundamental domains of $\widetilde{\mathcal{W}}, \widetilde{\Gamma}$, respectively): $\lim_{r\to\infty} \lambda(U^r) = \lambda(R)$.

Now, let Π be the full-rank lattice that is, by assumption, contained in $\Gamma \cap \widetilde{\Gamma}$, and let Ω denote its compact fundamental domain. Note that, since Π is a sublattice of both translation lattices, we may reduce the sets U^r modulo Π while maintaining all three of the properties mentioned above. This is the most important step in this second limiting process. So we obtain a new family of sets,

$$V^r = r_\Pi(U^r).$$

Now the family V^r is completely contained within the Hausdorff-metric space $\mathcal{K}(\Omega)$ which, like $\mathcal{K}(r)$ earlier, is complete and totally bounded, hence sequentially compact. Thus, if $(r_n)_{n=1}^\infty$ is a sequence of positive reals tending to infinity, then it is immediate that the sequence $(V^{r_n})_{n=1}^\infty$ of sets in $\mathcal{K}(\Omega)$ has a convergent subsequence. Let $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$ be its limit. We claim that $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$ is a two-way tiling set for $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$, as desired.

First, $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$ is $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-compatible. This can be shown by a contradiction argument even easier than the similar one at the end of the first limiting process – this time, all the elements of the convergent sequence are already $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-compatible.

Finally, since $\lambda(V^{r_n}) \to \lambda(R)$ as $n \to \infty$ and Lebesgue measure is continuous with respect to Hausdorff limits (*cf.* Def. 2.9, property (2)), we have $\lambda(W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}) = \lambda(R)$, i.e. $W_{\widetilde{\mathcal{W}},\widetilde{\Gamma}}$ is of full measure. This completes the proof of Proposition 3.9, and hence the proof of Theorem 3.1'. □

3.10 Remark. So far we have only proved the weaker Theorem 3.1', we will now explain how this can be generalised to yield Theorem 3.1. The modification affects only sections 3.2.3 and 3.2.4.

Our proof relied on making a certain restriction on the lattice $\widetilde{\Gamma}$. In section 3.2.3, we have seen that if $\widetilde{\Gamma} = D\Gamma$, where D is a diagonal matrix with irrational entries, the reduction $r_\Gamma(\widetilde{\Gamma})$ is dense in the fundamental domain of Γ. This density result holds for more general matrices M linking Γ and $\widetilde{\Gamma}$; in particular, it still holds true if M is a matrix with precisely one irrational entry in every row and every column, and rational entries everywhere else (obviously, the case of M being diagonal is an instance of this).

Now, in the first limiting operation in section 3.2.4, we defined a sequence of lattices $\widetilde{\Gamma}_n = \beta_n\widetilde{\Gamma}$ which, in a sense, "converges to $\widetilde{\Gamma}$ from below" (in terms of measure of fundamental domains), each such matrix being diagonal with irrational entries. This can also be generalised, albeit by use of brute force: We may instead define a sequence $\widetilde{\Gamma}_n$ such that the column vectors of $\widetilde{\Gamma}_n$ uniformly converge to those of $\widetilde{\Gamma}$, each matrix $\widetilde{\Gamma}_n$ having an irrational entry in every row and every column, and finally, such that $|\det \widetilde{\Gamma}_n|$ converges to $|\widetilde{\mathcal{W}}|^{-1}$ from below. When reviewing the "compatibility proof" illustrated in Figure 3.1 with these modifications made, the result carries through. The second limiting process remains unaffected.

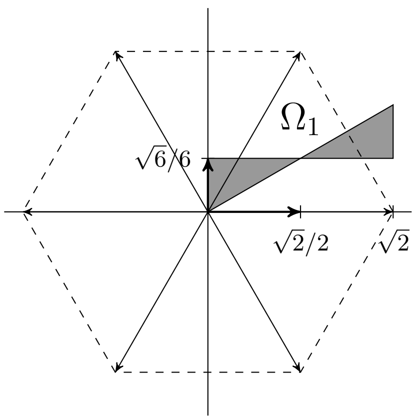
This way, our main result is proved.

As a concluding remark, we note the similarity between results 1.23 and 3.9, both being congruency theorems for a dual dynamical system which are proved by iterative methods. Theorem 3.9 is related to Ergodic Theory, i.e. the study of measure preserving transformations of probability spaces (*cf.* [Fur99], and chapter 2 of [Gab07]).

3.3 Examples of triple wavelet sets in two dimensions

In the following table we list a few examples of reflection-translation tiling sets $\Omega_i \subset \mathbb{R}^2$, $i = 1, \ldots, 6$, each of which tiles $\mathbb{R}^2$ under a corresponding pair $(\widetilde{\mathcal{W}_i}, \widetilde{\Gamma}_i)$ with $\widetilde{\mathcal{W}_i} = \mathcal{W}_i \ltimes \Gamma_i$. With an application of Lemma 3.6, these sets give rise to three-way tiling sets under any dilation group. Note that, unlike the existence proof in section 3.2, the proof of Lemma 3.6 is constructive, so that the sets given below may be used to calculate triple wavelet sets explicitly.

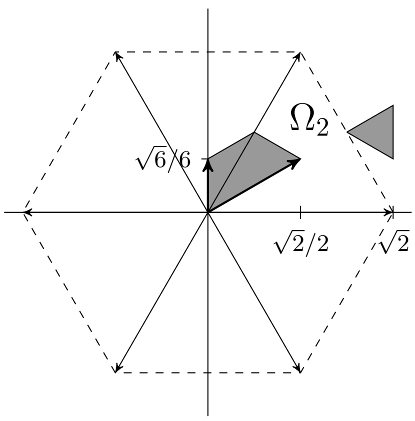
The sets Ω_4 and Ω_6 were found by the author, the other sets are taken from [LMÓ07].



$$\mathcal{W}_1 = A_2$$

$$\Gamma_1 = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix} \mathbb{Z}^2$$

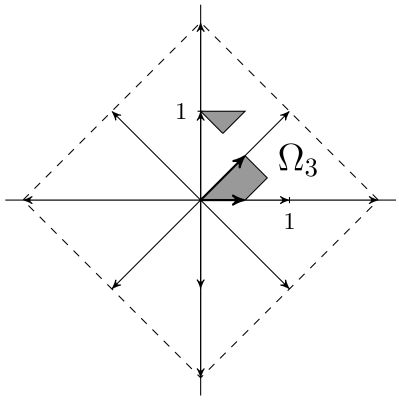
$$\widetilde{\Gamma}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{6}}{6} \end{pmatrix} \mathbb{Z}^2$$



$$\mathcal{W}_2 = A_2$$

$$\Gamma_2 = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix} \mathbb{Z}^2$$

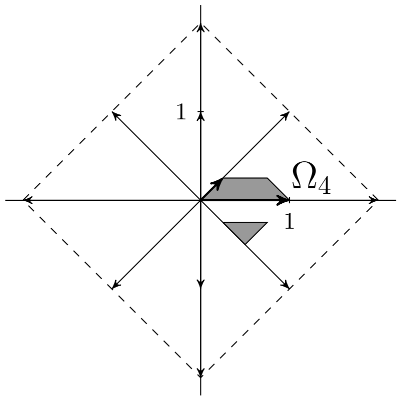
$$\widetilde{\Gamma}_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \end{pmatrix} \mathbb{Z}^2$$



$$\mathcal{W}_3 = B_2$$

$$\Gamma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2$$

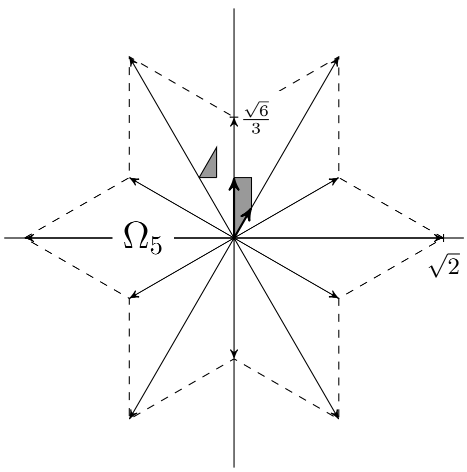
$$\widetilde{\Gamma}_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$$



$$\mathcal{W}_4 = B_2$$

$$\Gamma_4 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2$$

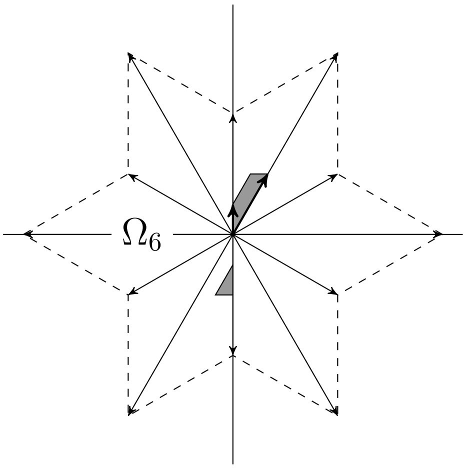
$$\widetilde{\Gamma}_4 = \begin{pmatrix} 1 & \frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix} \mathbb{Z}^2$$



$$\mathcal{W}_5 = G_2$$

$$\Gamma_5 = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{6} \end{pmatrix} \mathbb{Z}^2$$

$$\widetilde{\Gamma}_5 = \begin{pmatrix} \frac{\sqrt{2}}{12} & 0 \\ \frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{6} \end{pmatrix} \mathbb{Z}^2$$



$$\mathcal{W}_6 = G_2$$

$$\Gamma_6 = \begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{6} \end{pmatrix} \mathbb{Z}^2$$

$$\widetilde{\Gamma}_6 = \begin{pmatrix} \frac{\sqrt{2}}{6} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{12} \end{pmatrix} \mathbb{Z}^2$$

TABLE 3.1.

By Lemma 3.6, proving that these sets indeed give rise to three-way tiling sets (with any matrix dilation group) has been reduced to showing that each set Ω_i is a two-way tile for the associated pair $(\widetilde{\mathcal{W}}_i, \widetilde{\Gamma}_i)$, and that these pairs are commensurable. The latter property holds true since, for each i, in fact we have $\Gamma_i \subseteq \widetilde{\Gamma}_i$, whence immediately $\Gamma_i \cap \widetilde{\Gamma}_i \supseteq \Gamma_i$.

To show that Ω_i is a two-way tile for $(\widetilde{\mathcal{W}}_i, \widetilde{\Gamma}_i)$, we need to show that it is both $\widetilde{\mathcal{W}}_i$-congruent to a $\widetilde{\mathcal{W}}_i$-fundamental domain and $\widetilde{\Gamma}_i$-congruent to a $\widetilde{\Gamma}_i$-fundamental domain. We will just do this for Ω_4, the other sets are very similar in this respect.

Firstly, to obtain a $\widetilde{\mathcal{W}}$-tile, we take the triangle with vertices $(\frac{1}{4}, -\frac{1}{4})^{\mathsf{T}}, (\frac{1}{2}, -\frac{1}{2})^{\mathsf{T}}$, $(\frac{3}{4}, -\frac{1}{4})^{\mathsf{T}}$, and reflect it along the x-axis, which is obviously an operation in $\widetilde{\mathcal{W}}$. Hence we obtain the triangle $(0,0)^{\mathsf{T}}, (1,0)^{\mathsf{T}}, (\frac{1}{2}, \frac{1}{2})^{\mathsf{T}}$, which is a $\widetilde{\mathcal{W}}$-tile.

Secondly, for a $\widetilde{\Gamma}$-tile, we start with the same triangle and shift it by the vector $(\frac{1}{2}, \frac{1}{2})^{\mathsf{T}} \in \widetilde{\Gamma}$ to obtain the parallelepiped $(1,0)^{\mathsf{T}} \wedge (\frac{1}{4}, \frac{1}{4})^{\mathsf{T}}$, as desired.

Fig. 3.3 visualises these transformations.

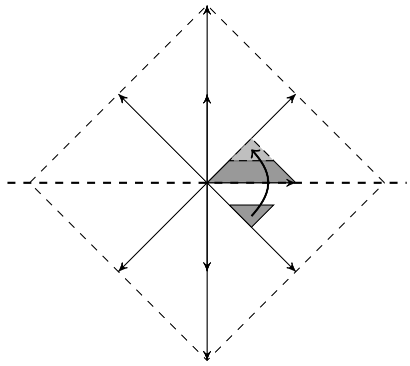


FIGURE 3.3.

3.4 Relation to third Hilbert Problem

As mentioned earlier (following the statement of Theorem 3.1'), our construction of three-way tiling sets in arbitrary dimensions is inadequate for producing concrete examples by explicit computation due to the infinitesimal processes involved; it is not possible to even vaguely guess their appearance. The examples of triple wavelet sets in two dimensions that we have seen are straightforward and easily understood by looking at their depictions, whereas it is conceivable that, by contrast, triple wavelet sets in higher dimensions will inevitably be fractal-like.

More precisely, the sets seen in the last section were simple as they were obtained by cutting the translation (or reflection) fundamental domains along straight lines into *finitely many* subsets, and mapping each such piece to a different position by means of the group actions. Whether this can also be done in higher dimensions will be investigated in this section.

3.4.1 Scissors Congruences

The problem whether a polyhedron can be cut along hyperplanes and reassembled to a different polyhedron of the same volume has been studied under the name of "scissors congruences", where two polyhedra in $\mathbb{R}^n$ are said to be *scissors congruent* if they can be transformed into each other in this way. It received particular attention in 1900 when Hilbert included it among his famous list of 23 problems as the third one (although, of course, no order of priority was implied). The precise way it was formulated is as follows:

Hilbert's 3rd problem. *Given two tetrahedra of the same base area and the same height, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?*
This problem was the first of the list to be solved: Hilbert's student Max Dehn gave a negative answer only a year later by producing a counterexample [Deh01]. His method relied on using what became known as the *Dehn invariants* of polyhedra, which are preserved by scissors congruences. Since scissors congruence is an equivalence relation, all polyhedra of an equivalence class have the same Dehn invariant, so it sufficed to find two polyhedra (in Dehn's case, tetrahedra) of different such invariants.
Note that in dimension 2, it had long been known that two polygons are always scissors congruent, this is known as the Bolyai-Gerwien Theorem. Subsequent work showed, however, that in any dimension $d > 2$ there exist pairs of polyhedra that are scissors incongruent.

Returning to reflection-translation tiles, it is clear that if the respective fundamental domains of a Weyl group and a lattice (being polyhedra) have different Dehn invariants and are hence scissors incongruent, a two-way tile (and hence also a triple wavelet set) cannot consist of a finite union of polyhedra, and may thus be more difficult to construct than if the fundamental domains were scissors congruent. In the following we will try to determine for which pairs $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$ we inevitably obtain such complicated sets – note that a fractal-like set is also more unwieldy for computing associated wavelet bases than sets consisting of finite unions of polyhedra.

3.4.2 Mürner's Results

Let $\widetilde{\mathcal{W}}$ be an affine Weyl group, and $\widetilde{\Gamma}$ a translation lattice. As in section 3.2, we denote by R the fundamental domain of $\widetilde{\mathcal{W}}$, and by T the fundamental domain of $\widetilde{\Gamma}$. Let $\widetilde{\mathcal{W}}$ and $\widetilde{\Gamma}$ be such that $\lambda(R) = \lambda(T)$. We denote scissors congruence by $\sim_s$.

To anticipate a result: It is actually true that $R \sim_s T$ for all Weyl groups and translation lattices. Note however that showing this is by far not enough to show that there exist $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-tiles consisting of a union of polyhedra: Scissors congruence, in general, allows the constituent pieces of a polyhedron to be moved around arbitrarily to yield another, i.e. any translations and rotations are allowed. However, a PhD thesis by em. Prof. Peter Mürner [Mür77] has discussed congruence relations where

the allowable transformations ("Bewegungsgruppen") are restricted.
In fact, this work deals specifically with transformation groups very similar to affine Weyl groups: The groups considered consist of a product of a finite rotation or reflection group, and the group of all translations by vectors in $\mathbb{R}^n$. Heightening the relevance of this text even more, the congruence properties of tiling polyhedra are specifically addressed.

To be more precise, let $G = D \ltimes \mathbb{R}^n$ be the group described above (in the notation of [Mür77] – i.e., D is not a dilation group as before, but a finite group of isometries of $\mathbb{R}^n$ fixing the origin). Let $\sim_G$ denote the relation of *scissors congruence via elements of* G. Following further the notation of this author, we denote by $\underline{P}^d_G$ all polyhedra in $\mathbb{R}^d$ that are G-tiles, and by $\underline{W}^d_G$ all polyhedra that are G-scissors congruent to a d-cube. Mürner shows the following theorem:

3.11 Theorem (Theorem 2 of [Mür77])**.** *The inclusion*

$$\underline{P}^d_G \subset \underline{W}^d_G$$

holds for all G *as above and* $d \in \mathbb{N}$.

This says, in other words, that any G-tile is G-congruent to a cube.

In the same text, an analogous theorem from an earlier work relating to translation tiles is also quoted. Replacing the group G by the group of ordinary translations, let $\underline{P}^d_T$ be the set of all polyhedra tiling $\mathbb{R}^d$ under translations, and $\underline{W}^d_T$ the set of all polyhedra that are scissors congruent to a cube via translations. We have the result

3.12 Theorem. *The inclusion*

$$\underline{P}^d_T \subset \underline{W}^d_T$$

holds in all dimensions d.

To interpret these two results for a $\widetilde{\mathcal{W}}$-tile R and a $\widetilde{\Gamma}$-tile T: The first result implies that R is scissors congruent to a cube via transformations of the group $G = \mathcal{W} \ltimes \mathbb{R}^n$, and so is the set T by the second result (since the group of all translations is a subgroup of G). This implies that $T \sim_G R$.

The one weakness of these results is that only the continuous group of translations is considered, which the author has not succeeded to resolve. If, in the above results, G could actually be replaced by an affine Weyl group, then this would immediately show the existence of polyhedral $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$-tiles for many pairs $(\widetilde{\mathcal{W}}, \widetilde{\Gamma})$. However, we believe that the results presented already give a strong indication that this might be possible for some pairs of reflection and translation groups.

Chapter 4

Oriented Oscillatory Waveforms

In this chapter we will briefly summarise some recent advances in the field of oriented oscillatory waveforms that involve reflection groups.
There exists a tremendous amount of literature on wavelet or frame systems that model the specific "directional shape" of a signal in time domain, well-known examples being *curvelets* [CD02, CDDY06], *contourlets* [DV05], *shearlets* [GLL+05], and ridgelets [CD99]. What appears to be relatively uncommon is that such "x-lets" may be associated with an MRA, of which we mention two examples.

The first piece of work is a pre-preprint [Bla07], kindly provided to the author by Prof. Weiss, from one of his collaborators, Jeffrey Blanchard. A reason to include it here is that it brings together the concepts of MRAs, MSF wavelets, and reflection groups introduced in this thesis.
Another recent advancement is some work by J. Krommweh and G. Plonka who have constructed directional Haar framelets on triangles, aimed at applications in image analysis. These frames are also associated with an MRA. However, for lack of a stronger connection to the remainder of this thesis, we will not pursue this further.

4.1 MSF Composite Dilation Wavelets

In his preprint, J. Blanchard introduces systems of composite dilation wavelets which use three groups of transformations to generate a wavelet basis. In precise terms:

- Let $a \in GL(n, \mathbb{R})$ be an invertible matrix, with the dilation operator D_a given by $D_a f(\mathbf{x}) = |\det(a)|^{-\frac{1}{2}} f(a^{-1}\mathbf{x}))$, and $A = \{a^j : j \in \mathbb{Z}\}$ the associated group of dilations;

- let $B \subset GL(n, \mathbb{R})$ be a finite subgroup of $GL(n, \mathbb{R})$, with associated dilation operator $D_b f(\mathbf{x}) = f(b^{-1}\mathbf{x})$;
- let $\Gamma = c\mathbb{Z}^n$ be a full-rank lattice (with $c \in GL(n, \mathbb{R})$), and $T_\mathbf{k}$ the translation operator given by $T_\mathbf{k} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{k})$.

4.1 Definition. $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$ is an $AB\Gamma$ *composite dilation wavelet* if the system

$$\{D_a^j D_b T_\mathbf{k} \psi^l : j \in \mathbb{Z}, b \in B, \mathbf{k} \in \Gamma, l = 1, \ldots, L\}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

The use of the name "composite dilation wavelet" is clear from this definition: The three transformation groups may be considered as a conventional translation group together with a group which is the product of two, not necessarily commuting, groups of matrix dilations. It is immediate that reflection and rotation groups are candidates for the group B; indeed we will see that such a choice makes possible a strong result.

The generalisation of the definition of MSF wavelets to $AB\Gamma$ composite dilation wavelets is straightforward:

4.2 Definition. $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$ is an *MSF* $AB\Gamma$ *composite dilation wavelet* if there exist sets $R_1, \ldots, R_L \subset \mathbb{R}^n$ such that $\widehat{\psi}^l = |\det(c)|^{\frac{1}{2}} \chi_{R_l}$ for all $l = 1, \ldots, L$, and if the system $\{D_a^j D_b T_\mathbf{k} \psi^l\}$, as given in Definition 4.1, is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Further, multiresolution analyses may be generalised to the case of composite dilation wavelets. To this end, condition 3 in Definition 1.4 is simply replaced by the requirement that

$$f \in V_j \Leftrightarrow f(a\cdot) \in V_{j+1},$$

whereas condition 4 is replaced by the requirement that there exists a $\phi \in V_0$ such that

$$\{D_b T_\mathbf{k} \phi : b \in B, \mathbf{k} \in \Gamma\}$$

is an ONB for V_0.

The investigations of such wavelet systems culminate in a strong yet perhaps surprisingly simple result:

4.3 Theorem (Theorem 10 of the preprint)**.** *If B is any group with fundamental domain bounded by n hyperplanes through the origin and $\Gamma = c\mathbb{Z}^n$ is any full-rank lattice, then there exists an MRA, MSF, composite dilation wavelet for $L^2(\mathbb{R}^n)$.*

It can be verified that this condition on B is satisfied, in particular, by any Weyl group. The author feels that this theorem opens up an interesting new perspective on the use of reflection groups in the construction of wavelet systems.

In addition, however, we note that if the group B is taken to be a Weyl group and the associated coroot lattice is chosen for the translation group Γ, then the $AB\Gamma$-system is precisely the system of unitary operators employed in the construction of fractal wavelet bases in chapter 2. Moreover, with B, Γ defined this way, the definitions of multi-resolution analyses agree fully. Hence the following result is obvious:

4.4 Proposition. Let D_q be an operator that dilates by an integer $q > 1$, let $\widetilde{\mathcal{W}}$ be an affine Weyl group, and let $\boldsymbol{\psi} = (\psi^1, \ldots, \psi^s)$ be a set of fractal functions such that the system $\{D_q^k \mathcal{B}_{\boldsymbol{\psi}} | k \in \mathbb{Z}\}$ is an MRA orthonormal wavelet system, as constructed in section 2.3. Then $\{D_q^k \mathcal{B}_{\boldsymbol{\psi}} | k \in \mathbb{Z}\}$ is an MRA, $AB\Gamma$-composite dilation wavelet system, with $A = qI$, $B = \widetilde{\mathcal{W}}$, and $\Gamma = \Gamma$, the coroot lattice of $\widetilde{\mathcal{W}}$.

In other words, MRA fractal surface wavelets are just a special case of MRA $AB\Gamma$ composite dilation wavelets. This link is fascinating; however, we consider it unlikely that further investigations will produce fractal surface wavelets that are, additionally, MSF composite dilation wavelets.

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